Horizontal Notation for Sequences

	0	k	len(b)
b	<= sorted	>=	

Example of an assertion about an sequence b. It asserts that:

1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]

	0	h	k
b			

Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.

b[h .. k – 1] has k – h elements in it.

h h+1

(h+1) – h = 1

1

Developing Algorithms on Sequences

- Specify the algorithm by giving its precondition and postcondition as pictures.
- Draw the invariant by drawing another picture that "generalizes" the precondition and postcondition
  - The invariant is true at the beginning and at the end
- The four loop design questions
  1. How does loop start (how to make the invariant true)?
  2. How does it stop (is the postcondition true)?
  3. How does the body make progress toward termination?
  4. How does the body keep the invariant true?

2

Generalizing Pre- and Postconditions

- Dutch national flag: tri-color
  - Sequence of 0..n-1 of red, white, blue "pixels"
  - Arrange to put reds first, then whites, then blues

	0				n
pre: b	?				

(values in 0..n-1 are unknown)

	0			n
post: b	reds	whites	blues	

	0	j	k	l	n
inv: b	reds	whites	?	blues	

Make the red, white, blue sections initially **empty**:

- Range i..i-1 has 0 elements
- Main reason for this trick

Changing loop variables turns invariant into postcondition.

3

Generalizing Pre- and Postconditions

- Finding the minimum of a sequence.

	0		n
pre: b	?		and n >= 0

(values in 0..n are unknown)

	0		n
post: b	x is the min of this segment		

	0	j	n
inv: b	x is min of this segment	?	

pre: j = 0
post: j = n

(values in j..n are unknown)

- Put negative values before nonnegative ones.

	0		n
pre: b	?		and n >= 0

(values in 0..n are unknown)

	0	k	n
post: b	< 0	>= 0	

	0	k	j	n
inv: b	< 0	?	>= 0	

pre: k = 0, j = n
post: k = j

(values in k..j are unknown)

4

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

	h		k
pre: b	x	?	

- Swap elements of b[h..k] and store in j to truthify post:

	h	i	i+1	k
post: b	<= x	x	>= x	

change: b (h … k) 3 5 4 1 6 2 3 8 1

into: b (h, i, k) 1 2 1 3 5 4 6 3 8

or: b (h, i, k) 1 2 3 1 3 4 5 6 8

- x is called the pivot value
  - x is not a program variable
  - denotes value initially in b[h]

5

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

	h		k
pre: b	x	?	

- Swap elements of b[h..k] and store in j to truthify post:

	h	i	i+1	k
post: b	<= x	x	>= x	

	h	i		j	k
inv: b	<= x	x	?	>= x	

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

6

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])
Remember, slicing always copies the list!
We want to partition the **original** list

7

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= x			x	?			>= x		
	h		i	i+1			j		k
	1	2	3	1	5	0	6	3	8

8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= x		x	?			>= x		
h		i	i+1			j		k
1	2	3	1	5	0	6	3	8
h			i	i+1		j		k
1	2	1	3	5	0	6	3	8
h			i		j			k
1	2	1	3	0	5	6	3	8
h				i	j			k
1	2	1	0	3	5	6	3	8

9

Dutch National Flag Variant

- Sequence of integer values
  - 'red' = negatives, 'white' = 0, 'blues' = positive
  - Only rearrange part of the list, not all

pre: b — h … k: ?

post: b — h … k: < 0 | = 0 | > 0

inv: b — h, t, i, j, k: < 0 | ? | = 0 | > 0

pre: t = h,
i = k+1,
j = k
post: t = i

10

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		?			= 0		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3

11

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		?			= 0		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3
h		t		i		j		k
-1	-2	3	-1	0	0	0	6	3
h			t	i		j		k
-1	-2	-1	3	0	0	0	6	3
h			t		j			k
-1	-2	-1	0	0	0	3	6	3

12