

Recall: Horizontal Notation

0		k		len(b)
b	<= sorted	>=		

Example of an assertion about an sequence b. It asserts that:

1. $b[0..k-1]$ is sorted (i.e. its values are in ascending order)
2. Everything in $b[0..k-1]$ is \leq everything in $b[k..len(b)-1]$

0		h		k
b				

Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is $k - h$.

$b[h .. k - 1]$ has $k - h$ elements in it.

h	h+1
(h+1) - h = 1	

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Partition Algorithm Implementation

```

def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else: # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
        
```

	<= x	x	?	>= x
h	i	i+1	j	k
1	2	3	1	5
0	6	3	8	

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        if b[i+1] >= x:
            # Move to end of block.
            _swap(b,i+1,j-1)
            j = j - 1
        else: # b[i+1] < x
            _swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
        
```

	<= x	x	?	>= x
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1	2	3	1	5
0	6	3	8	

h	i	i+1	j	k
1	2	1	3	5
0	6	3	8	

h	i	j	k
1	2	1	0
3	5	6	3
8			

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Dutch National Flag Variant

- Sequence of integer values
 - 'red' = negatives, 'white' = 0, 'blues' = positive
 - Only rearrange part of the list, not all

pre:

	h		?		k
b					

post:

	h		< 0		= 0		> 0		k
b									

inv:

	h		t		i		j		k
b			< 0		? = 0		> 0		

pre: $t = h,$
 $i = k+1,$
 $j = k$
 post: $t = i$

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Dutch National Flag Algorithm

```

def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1; j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[t+1] < 0:
            swap(b,t+1,t)
            t = t+1
        elif b[t+1] == 0:
            i = i-1
        else:
            swap(b,t+1,j)
            i = i+1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i,j)
        
```

	< 0	?	= 0	> 0
h	t	i	j	k
-1	-2	3	-1	0
0	0	0	6	3

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Changing the Invariant

- Different invariants = different code
 - Need to change how we initialize, stop
 - Also need to change the body of the loop

pre:

	h		?		k
b					

post:

	h		< 0		= 0		> 0		k
b									

inv:

	h		i		t		j		k
b			< 0		= 0		?		> 0

pre: $t = h,$
 $i = h,$
 $j = k$
 post: $t = j+1$

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Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points"""
    t = h; i = h; j = k;
    # b[h..t-1] < b[i..t-1] = b[t..j] ?, b[j+1..k] >
    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # b[h..i-1] < b[i..j] = b[j+1..k] >
    return (i,j)
```

VS

```
def dnf(b, h, k):
    """Returns: partition points"""
    t = h; i = k+1; j = k;
    # b[h..t-1] < b[t..i-1] ?, b[i..j] = b[j+1..k] >
    while t < i:
        if b[t] < 0:
            swap(b,t,i)
            t = t+1
        elif b[t] == 0:
            i = i-1
        else:
            swap(b,t,j)
            i = i-1; j = j-1
    # b[h..i-1] < b[i..j] = b[j+1..k] >
    return (i,j)
```

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Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1; j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[t] < 0:
            swap(b,t,i)
            t = t+1
        elif b[t] == 0:
            i = i-1
        else:
            swap(b,t,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i,j)
```

< 0	$?$	$= 0$	> 0
h	t	i	j
-1	-2	3	-1
0	0	0	6
3	3	3	3

h	t	i	j	k
-1	-2	-1	3	3
0	0	0	0	6
3	3	3	3	3

h	t	j	k
-1	-2	-1	3
0	0	0	6
3	3	3	3

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Flag of Mauritius

- Now we have four colors!
 - Negatives: 'red' = odd, 'purple' = even
 - Positives: 'yellow' = odd, 'green' = even

pre: b [h | | | k]

post: b [<0 odd | <0 even | ≥0 odd | ≥0 even]

inv: b [<0, o | <0, e | ≥0, o | ? | ≥0, e]



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Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t
-1	-3	-2	-4	7
5	-5	-6	1	0
2	4	2	4	2

h	r	s	i	t	k
-1	-3	-5	-4	-2	5
7	-6	1	0	2	4

Need two swaps for two spaces

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Flag of Mauritius

< 0, o	< 0, e	?	≥ 0, e
h	r=s	i	t
-1	-3	-7	-4
-2	-6	-5	1
0	2	4	2

h	r=s	i	t	k
-1	-3	-7	-4	-2
-6	-5	1	0	2
4	2	4	2	4

BUT NOT ALWAYS!

Have to check if second swap is okay

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Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t
-1	-3	-2	-4	7
5	-5	-6	1	0
2	4	2	4	2

h	r	s	i	t	k
-1	-3	-5	-4	-2	5
7	-6	1	0	2	4

h	r	s	i	t	k
-1	-3	-5	-4	-2	5
7	5	1	0	2	4

See algorithms.py for Python code

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