Lecture 26

Sequence Algorithms (Continued)

Announcements for This Lecture

Lab/Finals

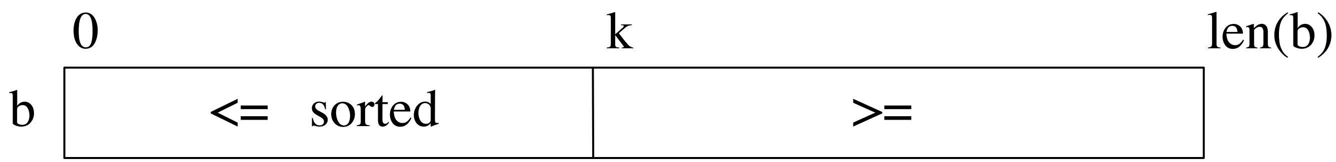
- **Lab 12** is the final lab
  - Can use Consulting hours
  - Due **next Wednesday 9:30**

- **Final: Dec 17th 9-11:30am**
  - Study guide is posted
  - Announce reviews next week.
- **Conflict with Final time?**
  - Submit to conflict to CMS **by next TUESDAY!**

Assignments

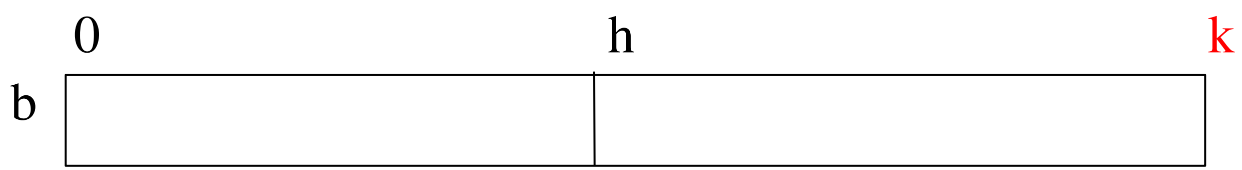
- **A6** is now graded
  - **Mean**: 89.5 **Median**: 93
  - **Std Dev**: 12.5
  - **Mean**: 15 hr **Median**: 15 hr
  - **Std Dev**: 7 hr
  - **SEVERAL** AI hearings
- **A7** is due **Tuesday Dec. 10**
  - Extensions are possible
  - Contact your lab instructor

Recall: Horizontal Notation

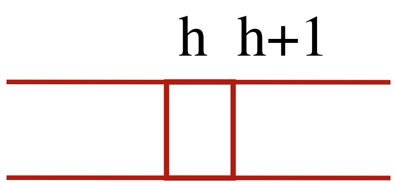


Example of an assertion about an sequence b. It asserts that:

1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.



(h+1) – h = 1

b[h .. k – 1] has k – h elements in it.

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

	h		k
pre: b	**x**	?	

- Swap elements of b[h..k] and store in j to truthify post:

	h	i	i+1	k
post: b	<= **x**	**x**	>= **x**	

---

	h	i		j	k
inv: b	<= **x**	**x**	?	>= **x**	

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
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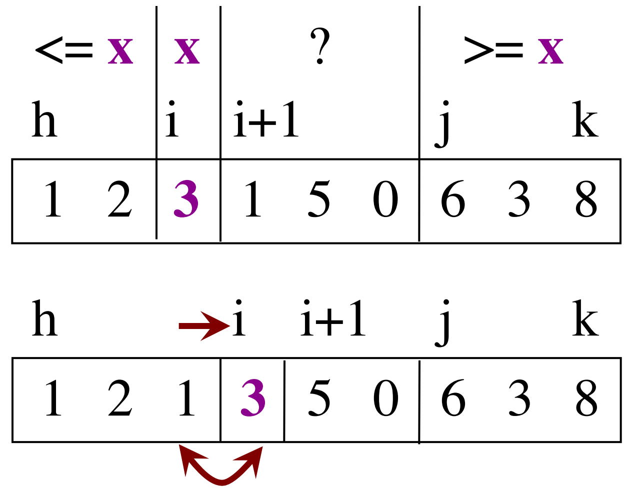

Partition Algorithm Implementation

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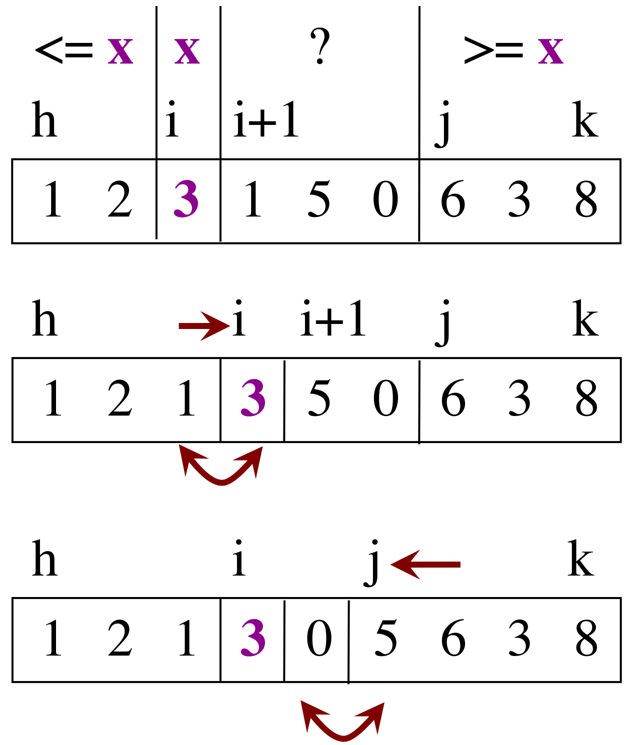

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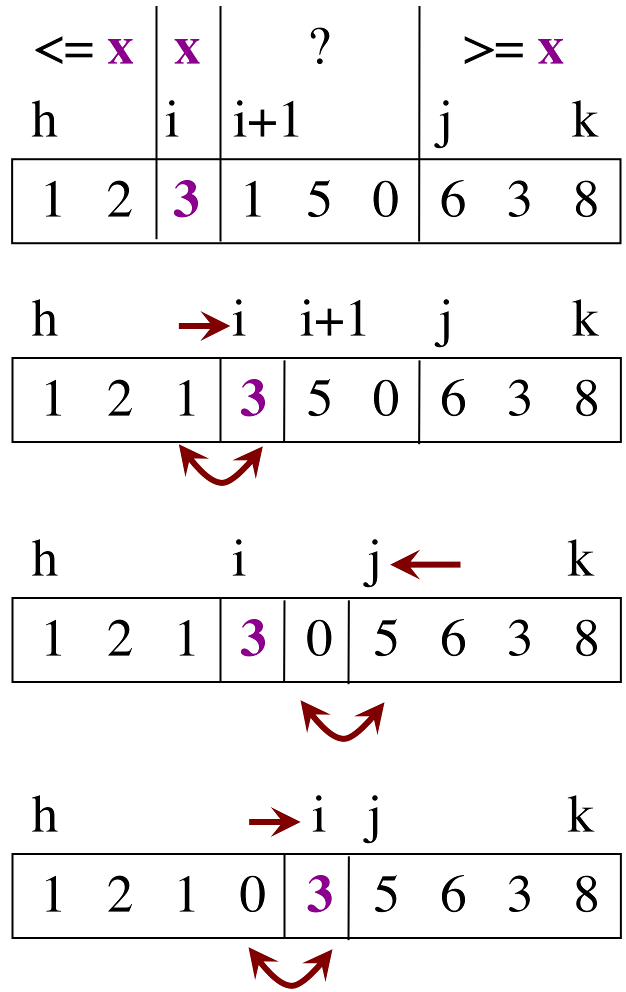
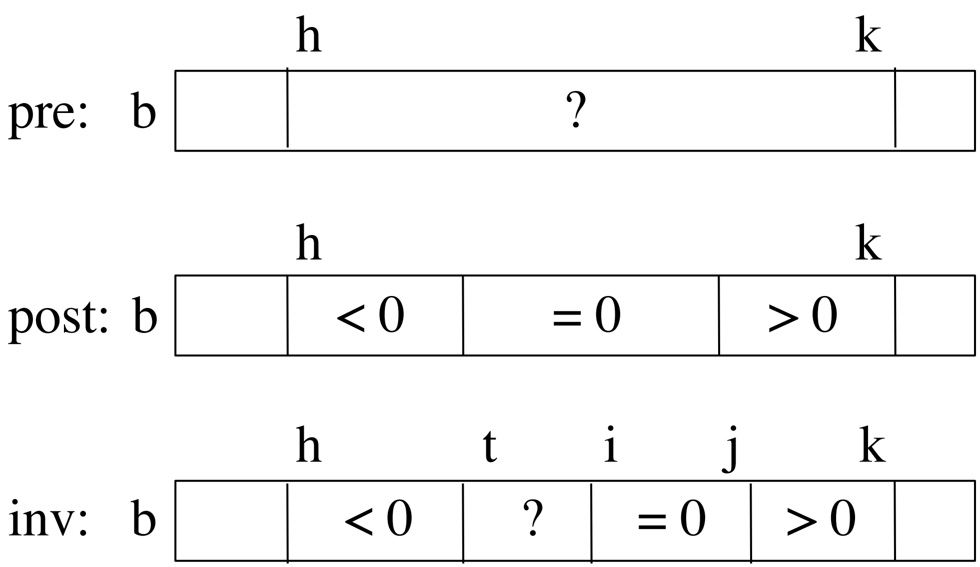


Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



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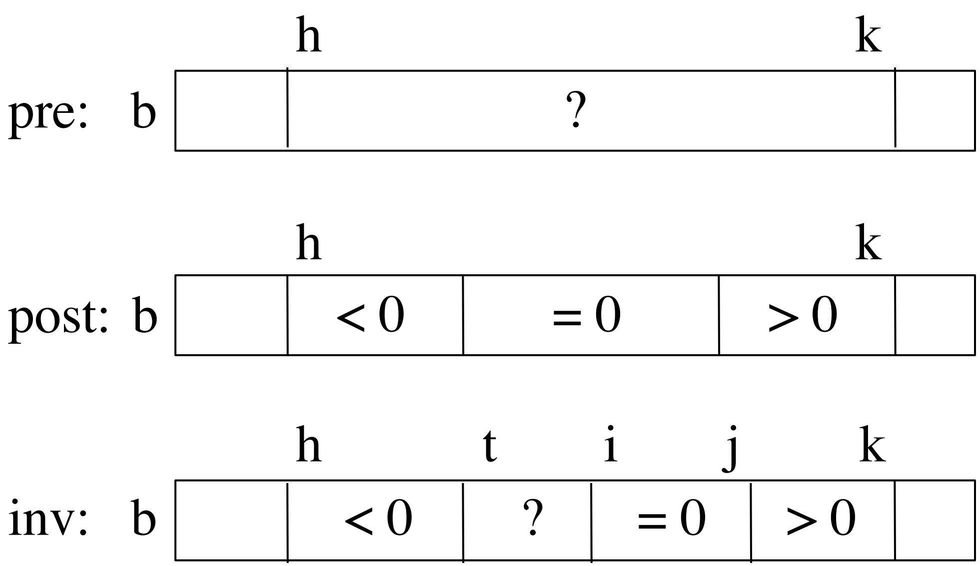

pre: t = h,
i = k+1,
j = k
post: t = i

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

Dutch National Flag Algorithm

```
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            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

h	t	i	j k
-1 -2	3 -1	0 0 0	6 3

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            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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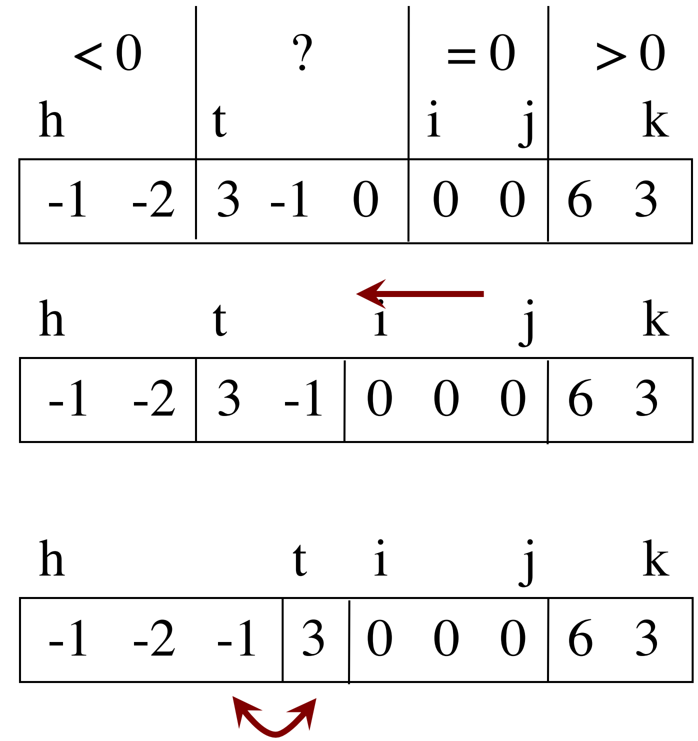

Dutch National Flag Algorithm

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    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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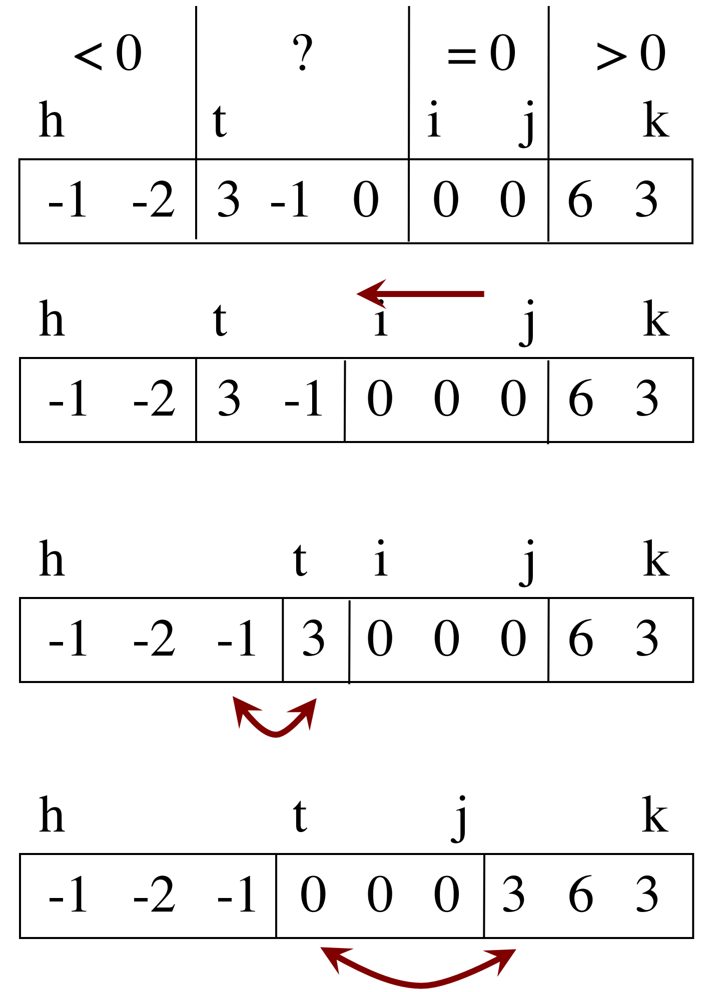
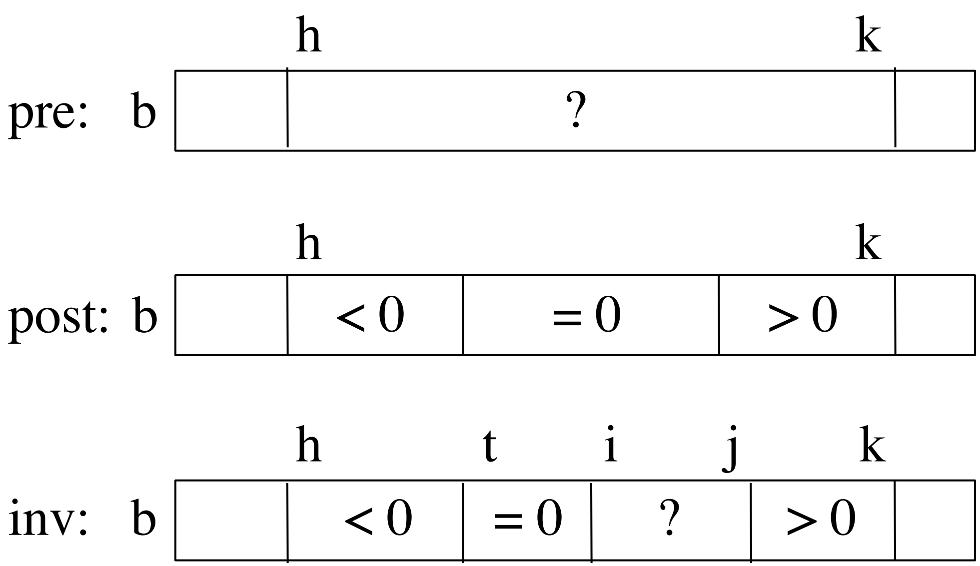
```



Changing the Invariant

- Different invariants = different code
  - Need to change how we initialize, stop
  - Also need to change the body of the loop



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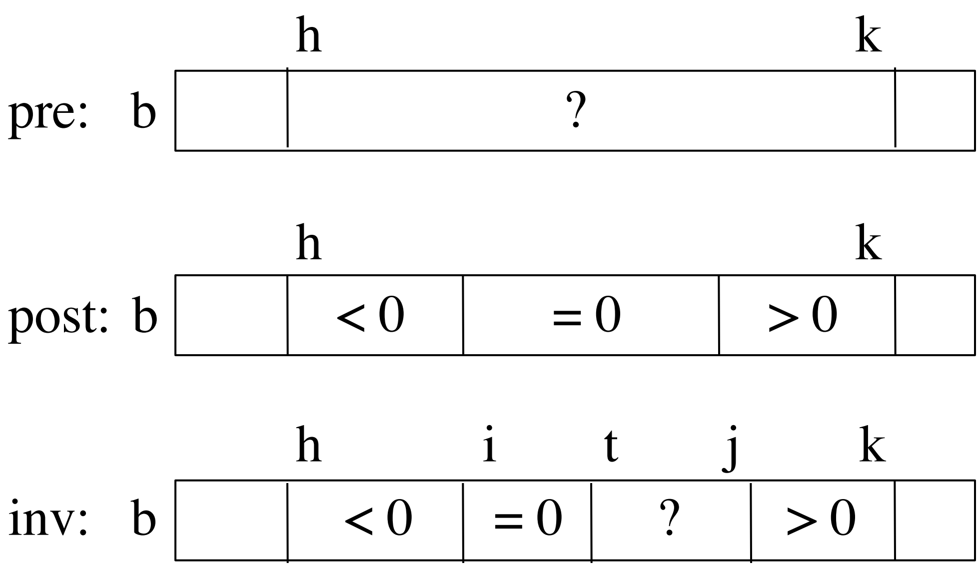

pre: t = h,
i = h,
j = k
post: **t = j+1**

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[???] < 0:
            ???

        elif b[???] == 0:
            ???
        else:
            ???

    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		= 0		?			> 0	
h		i		t		j		k
-1	-2	0	0	3	-1	0	6	3

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
            ???

    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		= 0		?			> 0	
h		i		t		j		k
-1	-2	0	0	3	-1	0	6	3

Changing the Invariant

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    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		= 0		?			> 0	
h		i		t		j		k
-1	-2	0	0	3	-1	0	6	3

h		i		t	j	k		
-1	-2	0	0	0	-1	3	6	3

Changing the Invariant

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def dnf(b, h, k):
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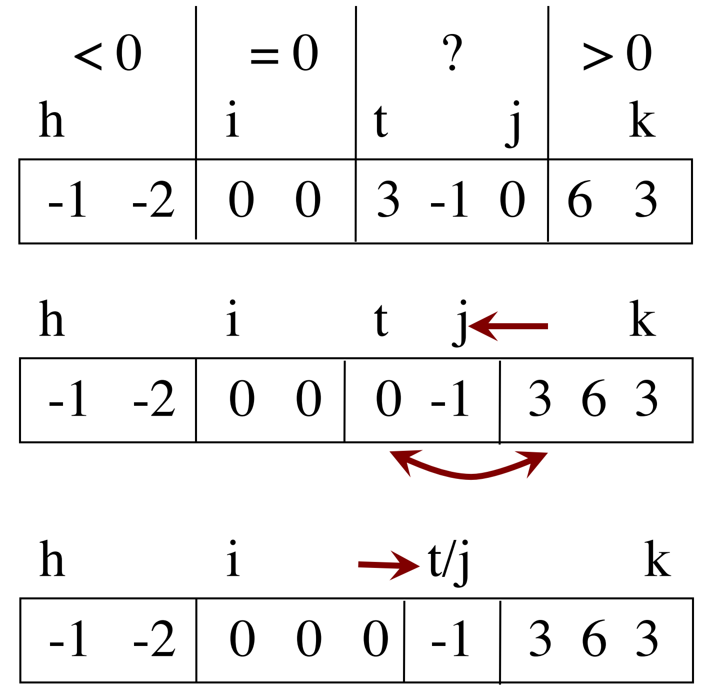
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



Changing the Invariant

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    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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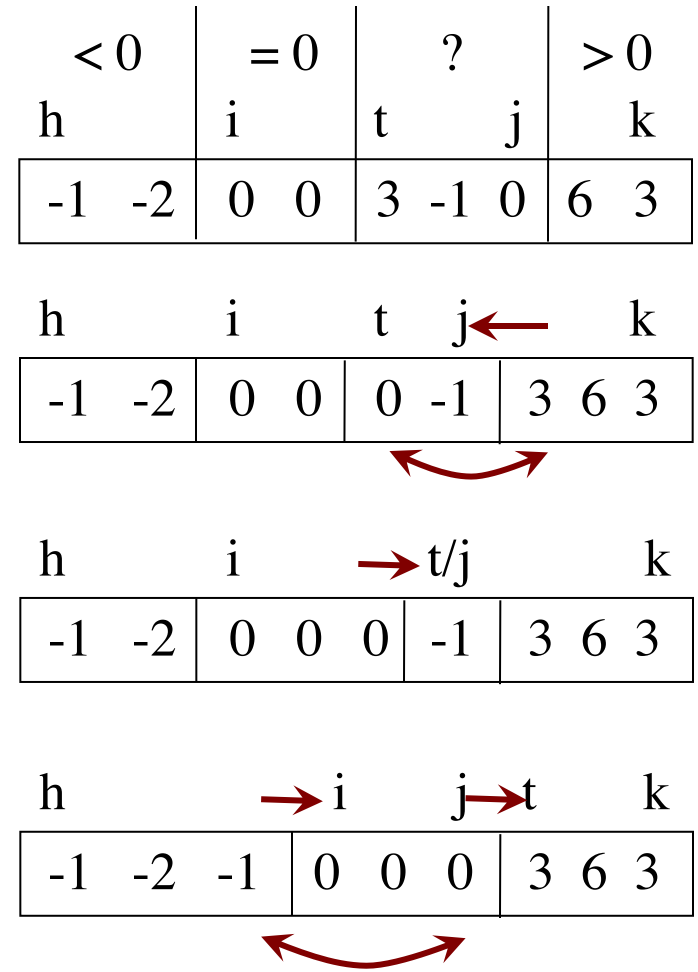
```



Changing the Invariant

```
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    t = h; i = h, j = k;
    # b[h..t-1] <, b[i..t-1] =, b[t..j] ?, b[j+1..k] >
    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

VS

```
def dnf(b, h, k):
    """Returns: partition points"""
    t = h; i = k+1, j = k;
    # b[h..t-1] <, b[t..i-1] ?, b[i..j] =, b[j+1..k] >
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

Flag of Mauritius

- Now we have four colors!
  - Negatives: ‘red’ = odd, ‘purple’ = even
  - Positives: ‘yellow’ = odd, ‘green’ = even

		h		k	
pre:	b		?		

		h				k	
post:	b		< 0 odd	< 0 even	≥ 0 odd	≥ 0 even	

		h	r	s	i	t	k
inv:	b		< 0, o	< 0, e	≥ 0, o	?	≥ 0, e

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	7	5	-2	-6	1	0	2	4

One swap is not good enough

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	-2	5	7	-6	1	0	2	4

Need two swaps for two spaces

Flag of Mauritius

< 0, o h		< 0, e r		≥ 0, o s		? i				≥ 0, e t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h			r		s		i			t	k
-1	-3	-5	-4	-2	5	7	-6	1	0	2	4

And adjust the loop variables

Flag of Mauritius

< 0, o			< 0, e			?			≥ 0, e	
h			r=s			i			t	k
-1	-3	-7	-4	-2	-6	-5	1	0	2	4

h			r=s			i			t	k
-1	-3	-7	-5	-2	-6	-4	1	0	2	4

BUT NOT ALWAYS!

Flag of Mauritius

< 0, o	< 0, e	?	≥ 0, e
h	r=s	i	t k
-1 -3 -7	-4 -2 -6	-5 1 0	2 4

h	r=s	i	t k
-1 -3 -7	-4 -2 -6	-5 1 0	2 4

BUT NOT ALWAYS!

Have to check if second swap is okay

Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

See `algorithms.py` for Python code

Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2 -6	7 5	1 0	2 4

See algorithms.py for Python code

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h			r		s		i			t	k
-1	-3	-5	-4	-2	5	7	-6	1	0	2	4

h			r			s		i		t	k
-1	-3	-5	-4	-2	-6	7	5	1	0	2	4

h			r			s			i	t	k
-1	-3	-5	-4	-2	-6	7	5	1	0	2	4

See algorithms.py for Python code

Extras Not Covered in Class

Loaded Dice

- Sequence p of length n represents n-sided die
  - Contents of p sum to 1
  - p[k] is probability die rolls the number k

1	2	3	4	5	6
0.1	0.1	0.1	0.1	0.3	0.3

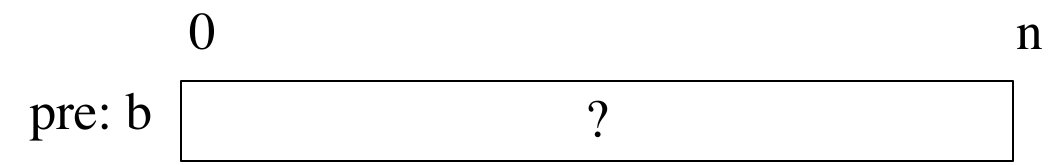
weighted d6, favoring 5, 6

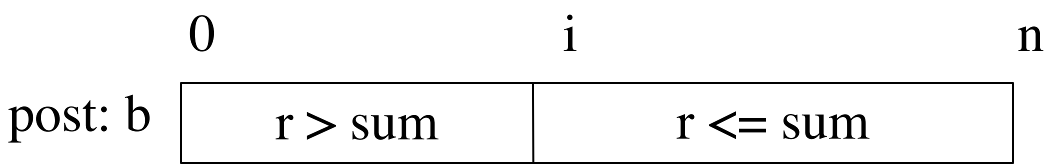
- Goal: Want to "roll the die"
  - Generate random number r between 0 and 1
  - Pick p[i] such that p[i-1] < r ≤ p[i]

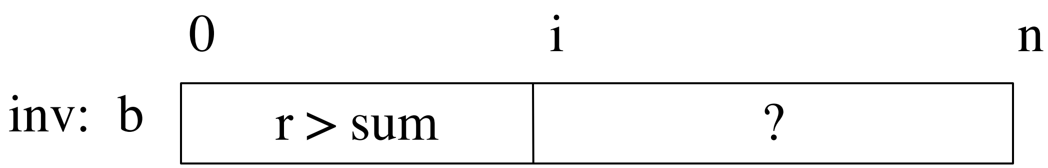
0.1	0.1	0.1	0.1	0.3	0.3
0.1	0.2	0.3	0.4	0.7	1.0

Loaded Dice

- **Want**: Value i such that p[i-1] < r <= p[i]







- Same as precondition if i = 0
- Postcondition is invariant + false loop condition

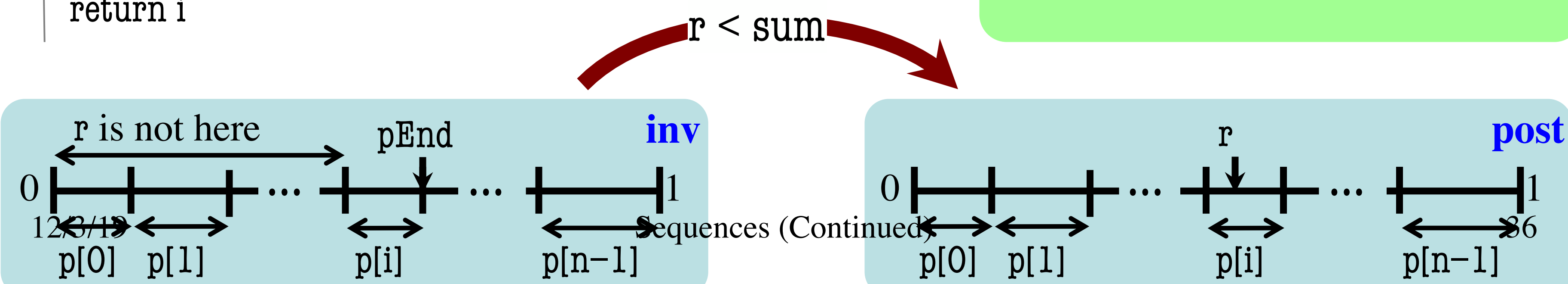
Loaded Dice

```
def roll(p):
    """Returns: randint in 0..len(p)-1; i returned with prob. p[i]
    Precondition: p list of positive floats that sum to 1."""
    r = random.random()     # r in [0,1)
    # Think of interval [0,1] divided into segments of size p[i]
    # Store into i the segment number in which r falls.
    i = 0;   sum_of = p[0]
    # inv: r >= sum of p[0] .. p[i–1]; pEnd = sum of p[0] .. p[i]
    while r >= sum_of:
        sum_of = sum_of + p[i+1]
        i = i + 1

    # post: sum of p[0] .. p[i–1] <= r < sum of p[0] .. p[i]
    return i
```

Analyzing the Loop

1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep **inv** true?



Reversing a Sequence

	h	k
pre: b	not reversed	

	h	k
post: b	reversed	

	h	k
change: b	1 2 3 4 5 6 7 8 9 9 9 9	
into b	9 9 9 9 8 7 6 5 4 3 2 1	

	h	i	j	k
inv: b	swapped	not reversed	swapped	