■ Previous Lecture:
  ■ Examples on vectors and simulation

■ Today’s Lecture:
  ■ Finite vs. Infinite; Discrete vs. Continuous
  ■ Vectors and vectorized code
  ■ Color computation with linear interpolation
  ■ plot and fill

■ Announcements:
  ■ Project 3 due tonight at 11pm
  ■ Prelim 1 on Tues 3/15 at 7:30pm
Loop patterns for working with a vector

%% Given a vector v

for k = 1:length(v)
    % Work with v(k)
    % E.g., disp(v(k))
end

%% Given a vector v

k = 1;
while k <= length(v)
    % Work with v(k)
    % E.g., disp(v(k))
    k = k+1;
end
Discrete vs. continuous

A plot is made from discrete values, but it can look continuous if there’re many points
Generating tables and plots

\[ x, y \] are vectors. A vector is a 1-dimensional list of values:

\[
x = \text{linspace}(0, 2\pi, 9);
y = \sin(x);
\text{plot}(x, y)
\]

<table>
<thead>
<tr>
<th>x</th>
<th>\sin(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.784</td>
<td>0.707</td>
</tr>
<tr>
<td>1.571</td>
<td>1.000</td>
</tr>
<tr>
<td>2.357</td>
<td>0.707</td>
</tr>
<tr>
<td>3.142</td>
<td>0.000</td>
</tr>
<tr>
<td>3.927</td>
<td>-0.707</td>
</tr>
<tr>
<td>4.712</td>
<td>-1.000</td>
</tr>
<tr>
<td>5.498</td>
<td>-0.707</td>
</tr>
<tr>
<td>6.283</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Note: \( x, y \) are shown in columns due to space limitation; they should be rows.
How did we get all the sine values?

Built-in functions accept arrays

<table>
<thead>
<tr>
<th>x</th>
<th>sin(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>1.57</td>
<td>1.0</td>
</tr>
<tr>
<td>3.14</td>
<td>0.0</td>
</tr>
<tr>
<td>4.71</td>
<td>-1.0</td>
</tr>
<tr>
<td>6.28</td>
<td>0.0</td>
</tr>
</tbody>
</table>

and return arrays

0.00 1.00 0.00 -1.00 0.00
After how many halvings will the disks disappear?
Xeno’s Paradox

- A wall is two feet away
- Take steps that repeatedly halve the remaining distance
- You never reach the wall because the distance traveled after n steps =

\[ 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n} \]
Example: “Xeno” disks

Draw a sequence of 20 disks where the \((k+1)\)th disk has a diameter that is half that of the \(k\)th disk.

The disks are tangent to each other and have centers on the \(x\)-axis.

First disk has diameter 1 and center \((1/2, 0)\).
Example: “Xeno” disks

What do you need to keep track of?

- Diameter (d)
- Position
  - Left tangent point (x)

<table>
<thead>
<tr>
<th>Disk</th>
<th>x</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0+1</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>0+1+1/2</td>
<td>1/4</td>
</tr>
</tbody>
</table>
% Xeno Disks

DrawRect(0,-1,2,2,'k')

% Draw 20 Xeno disks
% Xeno Disks

DrawRect(0,-1,2,2,'k')
% Draw 20 Xeno disks

for k= 1:20

end
% Xeno Disks

DrawRect(0,-1,2,2,'k')

% Draw 20 Xeno disks

d= 1;
x= 0;  % Left tangent point

for k= 1:20

end
% Xeno Disks

DrawRect(0,-1,2,2,'k')
% Draw 20 Xeno disks

d= 1;
x= 0; % Left tangent point

for k= 1:20
  % Draw kth disk

  % Update x, d for next disk

end
% Xeno Disks

DrawRect(0,-1,2,2,'k')
% Draw 20 Xeno disks

d= 1;
x= 0;  % Left tangent point
for k= 1:20
    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, 'y')
    % Update x, d for next disk
    x= x+d;
d= d/2;
end
Here’s the output… Shouldn’t there be 20 disks?

The “screen” is an array of dots called pixels.

Disks smaller than the dots don’t show up.

The 20th disk has radius < 0.00001
Fading Xeno disks

- First disk is yellow
- Last disk is black (invisible)
- **Interpolate** the color in between
Color is a 3-vector, sometimes called the RGB values

- Any color is a mix of red, green, and blue
- Example:
  
  \[
  \text{colr} = [0.4 \ 0.6 \ 0]
  \]

- Each component is a real value in \([0,1]\)
- \([0 \ 0 \ 0]\) is black
- \([1 \ 1 \ 1]\) is white
% Draw n Xeno disks

\[
d = 1; \\
x = 0; \quad \% \text{Left tangent point}
\]

\[
\text{for } k = 1:n \\
\text{% Draw kth disk}
\text{DrawDisk}(x+d/2, 0, d/2, 'y')
\text{x = x+d;}
\text{d = d/2;}
\text{end}
\]
% Draw n Xeno disks

d = 1;
x = 0;  % Left tangent point

for k = 1:n

    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, [1 1 0])
    x = x+d;
    d = d/2;

end

A vector of length 3
% Draw n fading Xeno disks

d = 1;
x = 0; % Left tangent point

yellow = [1 1 0];
black = [0 0 0];

for k = 1:n
    % Compute color of kth disk

    % Draw kth disk
    DrawDisk(x + d/2, 0, d/2, _______)

    x = x + d;
    d = d / 2;
end
Example: 3 disks fading from yellow to black

\[
\begin{align*}
    r &= 1; \quad \% \text{radius of disk} \\
    \text{yellow} &= [1 \ 1 \ 0]; \\
    \text{black} &= [0 \ 0 \ 0]; \\
    \% \text{Left disk yellow, at } x=1 \\
    \text{DrawDisk}(1,0,r,\text{yellow}) \\
    \% \text{Right disk black, at } x=5 \\
    \text{DrawDisk}(5,0,r,\text{black}) \\
    \% \text{Middle disk with average color, at } x=3 \\
    \text{colr} &= 0.5*\text{yellow} + 0.5*\text{black}; \\
    \text{DrawDisk}(3,0,r,\text{colr})
\end{align*}
\]
Example: 3 disks fading from yellow to black

\[
\begin{align*}
\text{r} &= 1; \quad \% \text{ radius of disk} \\
\text{yellow} &= [1 \ 1 \ 0]; \\
\text{black} &= [0 \ 0 \ 0]; \\
\% \text{ Left disk yellow, at x=1} \\
\text{DrawDisk}(1,0,\text{r,yellow}) \\
\% \text{ Right disk black, at x=5} \\
\text{DrawDisk}(5,0,\text{r,black}) \\
\% \text{ Middle disk with average color, at x=3} \\
\text{colr} &= 0.5 * \text{yellow} + 0.5 * \text{black}; \\
\text{DrawDisk}(3,0,\text{r,}\text{colr})
\end{align*}
\]
Example: 3 disks fading from yellow to black

\begin{verbatim}
\texttt{r= 1; \% radius of disk}
\texttt{yellow= [1 1 0];}
\texttt{black = [0 0 0];}
\texttt{\% Left disk yellow, at x=1}
\texttt{DrawDisk(1,0,r,yellow)}
\texttt{\% Right disk black, at x=5}
\texttt{DrawDisk(5,0,r,black)}
\texttt{\% Middle disk with average color, at x=3}
\texttt{colr= 0.5*yellow + 0.5*black;}
\texttt{DrawDisk(3,0,r,colr)}
\end{verbatim}

Vectorized addition

\begin{align*}
\begin{bmatrix}
0.5 & 0.5 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\end{align*}

= \begin{bmatrix}
0.5 & 0.5 & 0 \\
\end{bmatrix}
Vectorized code allows an operation on multiple values at the same time.

yellow = [1 1 0];
black = [0 0 0];

% Average color via vectorized op
colr = 0.5*yellow + 0.5*black;

% Average color via scalar op
for k = 1:length(black)
    colr(k) = 0.5*yellow(k) + 0.5*black(k);
end
% Draw n fading Xeno disks

d = 1;
x = 0;  % Left tangent point

yellow = [1 1 0];
black = [0 0 0];

for k = 1:n

    % Compute color of kth disk

    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, ______)
    x = x+d;
d = d/2;

end
% Draw n fading Xeno disks

d = 1;
x = 0; % Left tangent point
yellow = [1 1 0];
black = [0 0 0];

for k = 1:n
    % Compute color of kth disk
    colr = ___*black + ___*yellow;
    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, colr)
    x = x+d;
    d = d/2;
end
Use **linear interpolation** to obtain the colors. Each disk has a color that is a linear combination of yellow and black. Let $f$ be a fraction in $(0,1)$ ...

\[
f = \text{???} \]
\[
colr = f \times \text{black} + (1-f) \times \text{yellow};\]
Linear interpolation

<table>
<thead>
<tr>
<th>x</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>110</td>
</tr>
<tr>
<td>10</td>
<td>118</td>
</tr>
<tr>
<td>11</td>
<td>126</td>
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<tr>
<td>12</td>
<td>134</td>
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<td></td>
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</table>
### Linear interpolation

<table>
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<th>$x$</th>
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</tr>
<tr>
<td>10.25</td>
<td>?</td>
</tr>
<tr>
<td>10.50</td>
<td>?</td>
</tr>
<tr>
<td>10.75</td>
<td>?</td>
</tr>
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<td>134</td>
</tr>
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<td>:</td>
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\[
g(10.5) = \left[ g(11) + g(10) \right] / 2
\]
Linear interpolation

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</tbody>
</table>

\[
g(10.5) = \frac{1}{2} g(11) + \frac{1}{2} g(10)\]

\[
g(10.25) = \frac{1}{4}\cdot g(11) + \frac{3}{4}\cdot g(10)\]

\[
g(10.50) = \frac{2}{4}\cdot g(11) + \frac{2}{4}\cdot g(10)\]

\[
g(10.75) = \frac{3}{4}\cdot g(11) + \frac{1}{4}\cdot g(10)\]
Linear interpolation

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<td>12</td>
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</tbody>
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\[ g(10.5) = \frac{1}{2} g(11) + \frac{1}{2} g(10) \]

\[ g(10) = 0/4 \cdot g(11) + 4/4 \cdot g(10) \]
\[ g(10.25) = 1/4 \cdot g(11) + 3/4 \cdot g(10) \]
\[ g(10.50) = 2/4 \cdot g(11) + 2/4 \cdot g(10) \]
\[ g(10.75) = 3/4 \cdot g(11) + 1/4 \cdot g(10) \]
\[ g(11) = 4/4 \cdot g(11) + 0/4 \cdot g(10) \]

\[ f \cdot g(11) + (1-f) \cdot g(10) \]
% Draw n fading Xeno disks

d = 1;
x = 0; % Left tangent point
yellow = [1 1 0];
black = [0 0 0];

for k = 1:n

    % Compute color of kth disk
    f = ???
    colr = f*black + (1-f)*yellow;

    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, colr)
    x = x+d;
    d = d/2;
end
Linear interpolation

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<td>12</td>
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</tr>
<tr>
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<td>:</td>
</tr>
</tbody>
</table>

\[
g(10) = \frac{0}{4} \cdot g(11) + \frac{4}{4} \cdot g(10)
\]
\[
g(10.25) = \frac{1}{4} \cdot g(11) + \frac{3}{4} \cdot g(10)
\]
\[
g(10.50) = \frac{2}{4} \cdot g(11) + \frac{2}{4} \cdot g(10)
\]
\[
g(10.75) = \frac{3}{4} \cdot g(11) + \frac{1}{4} \cdot g(10)
\]
\[
g(11) = \frac{4}{4} \cdot g(11) + \frac{0}{4} \cdot g(10)
\]

\[
f \cdot g(11) + (1-f) \cdot g(10)
\]
% Draw n fading Xeno disks

d = 1;
xx = 0;  % Left tangent point
yellow = [1 1 0];
black = [0 0 0];

for k = 1:n

% Compute color of kth disk
f = ???
colr = f*black + (1-f)*yellow;

% Draw kth disk
DrawDisk(xx+d/2, 0, d/2, colr)
xx = xx+d;

d = d/2;

end
Rows of Xeno disks

for y = __ : __ : __

Code to draw one row of Xeno disks at some y-coordinate

end

Be careful with initializations
yellow=[1 1 0];  black=[0 0 0];

d= 1;

x= 0;

for k= 1:n
    % Compute color of kth disk
    f= (k-1)/(n-1);
    colr= f*black + (1-f)*yellow;
    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, colr)
    x=x+d;  d=d/2;
end
yellow = [1 1 0]; black = [0 0 0];

d = 1;
x = 0;

for k = 1:n
    f = (k-1)/(n-1);
    colr = f*black + (1-f)*yellow;
    DrawDisk(x+d/2, 0, d/2, colr);
    x = x + d;
    d = d/2;
end

Where to put the loop header for y = ___ : ___ : ___
yellow=[1 1 0]; black=[0 0 0];

for y = __:__:__
    d = 1;
    x = 0;
    initialization necessary for each row

for k = 1:n
    % Compute color of kth disk
    f = (k-1)/(n-1);
    colr = f*black + (1-f)*yellow;
    % Draw kth disk
    DrawDisk(x+d/2, 0, d/2, colr)
    x = x + d;
    d = d/2;
end
end
Does this script print anything?

```matlab
k = 0;
while 1 + 1/2^k > 1
    k = k+1;
end
disp(k)
```
Suppose you have a calculator with a window like this:

\[ + \ 2\ 4\ 1\ -\ 3 \]

representing \( 2.41 \times 10^{-3} \)
Floating point addition

\[ +241 - 3 \]

\[ +100 - 3 \]

Result: \[ +341 - 3 \]
Floating point addition

Result: 

+ 2 4 1 - 3

+ 1 0 0 - 4

+ 2 5 1 - 3
Floating point addition

\[ + 2 4 1 - 3 \]

\[ + 1 0 0 - 5 \]

Result: \[ + 2 4 2 - 3 \]
Floating point addition

\[ + \begin{array}{cccc} 2 & 4 & 1 \\ \end{array} - \begin{array}{c} 3 \\ \end{array} \]

\[ + \begin{array}{cccc} 1 & 0 & 0 \\ \end{array} - \begin{array}{c} 6 \\ \end{array} \]

Result: \[ + \begin{array}{cccc} 2 & 4 & 1 \\ \end{array} - \begin{array}{c} 3 \\ \end{array} \]
Floating point addition

Result: 

Not enough room to represent .002411
The loop DOES terminate given the limitations of floating point arithmetic!

```matlab
k = 0;
while 1 + 1/2^k > 1
    k = k+1;
end
disp(k)
```

1 + 1/2^53 is calculated to be just 1, so “53” is printed.
Patriot missile failure

In 1991, a Patriot Missile failed, resulting in 28 deaths and about 100 injured. The cause?

0.1

www.namsa.nato.int/gallery/systems
Inexact representation of time/number

- System clock represented time in tenths of a second: a clock tick every $1/10$ of a second

- Time = number of clock ticks $\times 0.1$

```
.00011001100110011001100110011...
```

- Value in Patriot system

```
.00011001100110011001100110011...
```

- Error of $0.000000095$ every clock tick
Resulting error

... after 100 hours

\[0.000000095 \times (100 \times 60 \times 60)\]

0.34 second

At a velocity of 1700 m/s, missed target by more than 500 meters!
Computer arithmetic is *inexact*

- There is error in computer arithmetic—floating point arithmetic—due to limitation in “hardware.” Computer memory is *finite*.

- What is $1 + 10^{-16}$?
  - $1.0000000000000001$ in real arithmetic
  - $1$ in floating point arithmetic (IEEE)

- Read Sec 4.3