$\boxed{\text{5 March 2021}}$ The Fast Fourier Transform (§5.6)

Announcements.

1. Prof. Kleinberg extra office hour today 11-12. (My office hour link is on the course website.)

2. Homework 4 (one problem) will be released within 24 hours, due Friday 3/12.

3. March 9-10 wellness days. Note office hour changes on the Google Calendar on 4820 website.

We saw a reduction from multiplying integers to multiplying polynomials with integer coefficients.

Karatsuba's Algorithm accomplishes both in $O(n^{\log_2(3)}) \approx O(n^{1.58})$.

Most recent discovery of a faster (asymptotically) integer multiplication algorithm.....

2019: Harvey & van der Hoeven announced a $O(n \log n)$ multiplication alg.
Under a well-known conjecture in information theory, this running time is optimal up to constant factors.

At the heart of this algorithm is the FFT; which can be thought of as a method for multiplying polynomials using $O(n \log n)$ arithmetic operations.

Multiplying polynomials is equivalent to <u>convolution of signals</u>.

A "signal" in this lecture is a sequence of numbers $a_0, a_1, \ldots, a_{n-1}$. (measurements in discrete time.)

Convolution takes

- signal $a_0, \ldots, a_{n-1}$
- "mask" $b_0, \ldots, b_{n-1}$

and outputs $c_0, \ldots, c_{2n-2}$ s.t.

$$c_k = \sum_{i+j=k} a_i b_j$$

Example: Modify a signal by averaging each number with its 2 neighbors.

a_0	a_1	a_2	a_3	a_4
5	9	-1	2	6

b_0	b_1	b_2	b_3	b_4
0	$1/3$	$1/3$	$1/3$	0

c_0	c_1	c_2	c_3	c_4	c_5	c_6
0	$5/3$	$14/3$	$\frac{5+9-1}{3}$	$\frac{9-1+2}{3}$	...	...

When we multiply polynomials

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

the coefficient of x^k in the polynomial $C(x) = A(x)\,B(x)$ is $c_k = \sum_{i+j=k} a_i b_j$.

So convolving signals is equivalent to multiplying polynomials.

Convolving in $O(n \log n)$ operations:

Use the fact that the coefficients of a polynomial of degree d are uniquely determined by its values at $d+1$ distinct points.

Recipe for multiplying $A(x) \cdot B(x)$.

(1) Let $d = \deg(A) + \deg(B) = \deg(A \cdot B)$. 1 op

(2) Select $d+1$ distinct numbers $z_0, \ldots, z_d$. $\emptyset$ ops

(3) Evaluate $A(z_0), \ldots, A(z_d)$ $O(n \log n)$

$B(z_0), \ldots, B(z_d)$

(4) Multiply value to compute

$$C(z_0) = A(z_0) \cdot B(z_0)$$

$$\vdots$$

$$C(z_d) = A(z_d) \cdot B(z_d)$$

$d+1$ mult.

(5) Use polynomial interpolation to find the coefficients of C, given values $C(z_0), \ldots, C(z_d)$. Inverse

FFT

= FFT, then divide by N.

Running time will depend on how fast we can implement steps 3 + 5.

$\longrightarrow A(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$

shows we can evaluate $A(z)$ for any z using $O(n)$ arithmetic ops.

Since $d = \deg(A) + \deg(B) = (n-1) + (n-1) = 2n-2$.

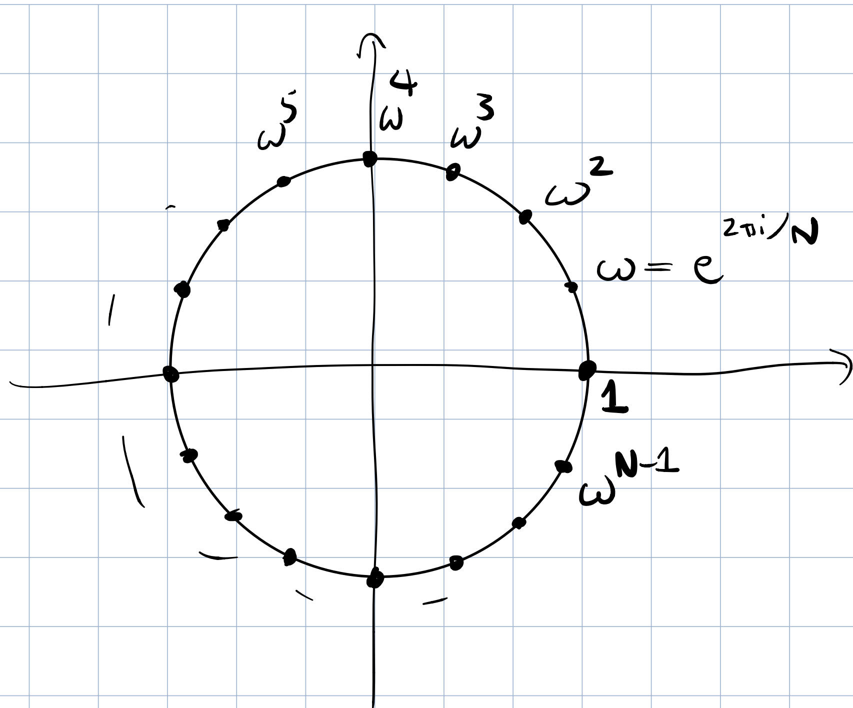
Evaluating $A(z_0), \dots, A(z_d)$ naively takes $O((d+1)\cdot n) = O(n^2)$ operations.

Divide-and-conquer with carefully chosen $z_0, \dots, z_d$ makes it faster.

Complex roots of unity: the numbers

$$e^{\frac{2\pi i k}{N}} \quad \text{for} \quad k = 0, 1, \dots, N-1$$

are called the complex N^{th} roots of unity. They are the solutions of $z^N = 1$ in the complex numbers.



Fourier transform of signal $a_0, a_1, \ldots, a_{N-1}$ is the sequence of complex numbers

$$A(1),\quad A(\omega),\quad A(\omega^2),\ \ldots,\ A(\omega^{N-1})$$

where $A(x) = a_0 + a_1 x + \ldots + a_{N-1} x^{N-1}$

and $\omega = e^{2\pi i/N}$.

When N is a power of 2 you can compute $A(1), A(\omega), \ldots, A(\omega^{N-1})$ in $O(N \log N)$ using divide and conquer. The FFT!

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}$$

$$= \underbrace{\boxed{\left(a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{N-2} x^{N-2}\right)}}_{A_{even}(x^2)}$$

$$+ \; x \cdot \underbrace{\boxed{\left(a_1 + a_3 x^2 + a_5 x^4 + \cdots + a_{N-1} x^{N-2}\right)}}_{A_{odd}(x^2)}$$

$$A(x) = A_{even}(x^2) + x \cdot A_{odd}(x^2)$$

As x runs through $1, \omega, \omega^2, \ldots, \omega^{N-1}$

x^2 runs through $\underline{1, \omega^2, \omega^4, \ldots, \omega^{2N-2}}$

This sequence is $1, \omega^2, \omega^4, \ldots, \omega^{N-2}$

repeated twice!

FFT: For $N = 2^k$, let $\omega = e^{2\pi i/N}$.

Given $A(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}$

1. Form

$$A_{even}(x) = a_0 + a_2 x + \cdots + a_{N-2} x^{(N-2)/2}$$

$$A_{odd}(x) = a_1 + a_3 x + \cdots a_{N-1} x^{(N-2)/2}$$

2. Recursively use FFT of order $\frac{N}{2}$ to evaluate A_{even}, A_{odd} at each of the points $1, \omega^2, \omega^4, \ldots, \omega^{N-2}$

3. For each $x \in \{1, \omega, \omega^2, \omega^3, \ldots, \omega^{N-1}\}$

$$A(x) = A_{even}(x^2) + x \cdot A_{odd}(x^2).$$

In $T(N)$ denotes # arithmetic ops to evaluate $A(1), A(\omega), \ldots, A(\omega^{N-1})$

Step 1. No arithmetic. Just splitting one array into 2 arrays.

Step 2. $2 \cdot T(\frac{N}{2})$ operations.

Step 3. N addition, N multiplication.

$$T(N) = 2 \cdot T\left(\frac{N}{2}\right) + 2N$$

$$\Longrightarrow \quad T(N) = O(N \log N).$$