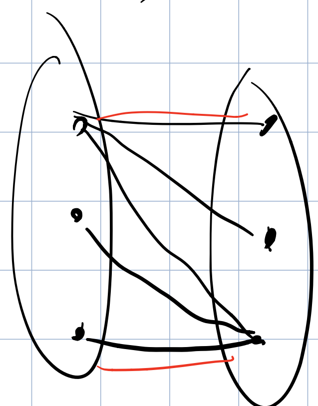
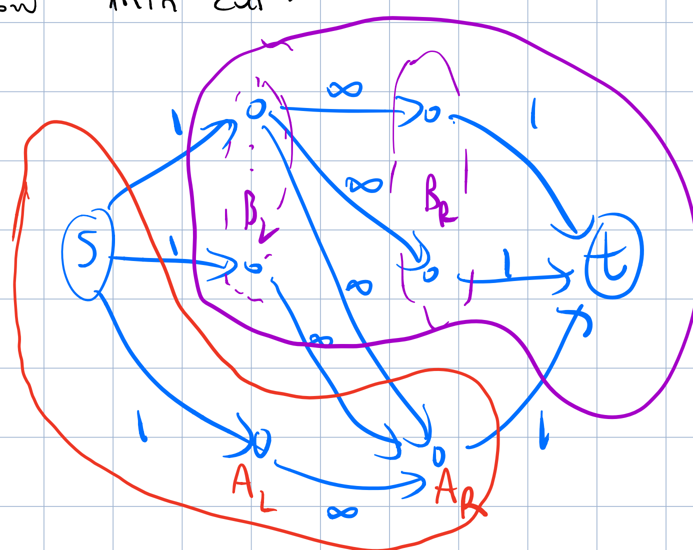
24 Mar 2021

More Flow Reductions

Announcement: Homework partner re-matching form
https://forms.gle/vdeCiE5NaPygkjpXA
Each group with ≥ 1 member filling out form will be split up and rematched using most recent form response of each member.

Interpreting max-flow min-cut:





A finite capacity cut is (A, B) s.t.

$$A = \{s\} \cup \{A_L\} \cup \{A_R\}$$
$$B = \{t\} \cup \{B_L\} \cup \{B_R\}$$

s.t. $\forall$ edges u,v in middle layer, it is not the case that $u \in A_L$, $v \in B_R$

The capacity of a cut with no edge from A_L to B_R is

It is the # of capacity 1 edges from A to B

$$= \#(A_R \to t \text{ edges}) + \#(s \to B_L \text{ edges})$$

$$= |A_R| + |B_L|.$$

Note. $A_R \cup B_L$ is a "vertex cover" of the bipartite graph, meaning every edge has at least one endpoint in $A_R \cup B_L$.

So max-flow min-cut theorem implies.....

König–Egerváry Theorem. If G is a finite bipartite graph, its maximum matching and its minimum vertex cover have the same number of elements.

Proof:

$$\begin{array}{ccc} \text{max matching} & = & v(\text{max flow}) \\ & & \| \\ \text{min vert cover} & = & \text{cap}(\text{min cut}) \end{array}$$

Baseball Elimination §7.12

Ex. 4 teams. Current # victories

NY	92
Balt	91
Tor	91
Boston	90

Remaining games: NY–Balt NY–Tor

Bos–Balt Bos–Tor

Balt–Tor

Can Boston finish the season at least tied for first place?

No: Bos 2 games behind NY.

To catch up, Boston must win twice, NY must lose twice.

$\Rightarrow$ one more victory for each of Baltimore, Toronto

$\Rightarrow$ the winner of the Balt–Toronto game will have 93 victories, ahead of Boston.

More succinct argument: Boston finishes with ≤ 92 wins.

If no other team gets > 92 wins, the remaining 3 teams collectively can get

≤ 2 victories. But they play 3 games against each other, each of which has a victor.

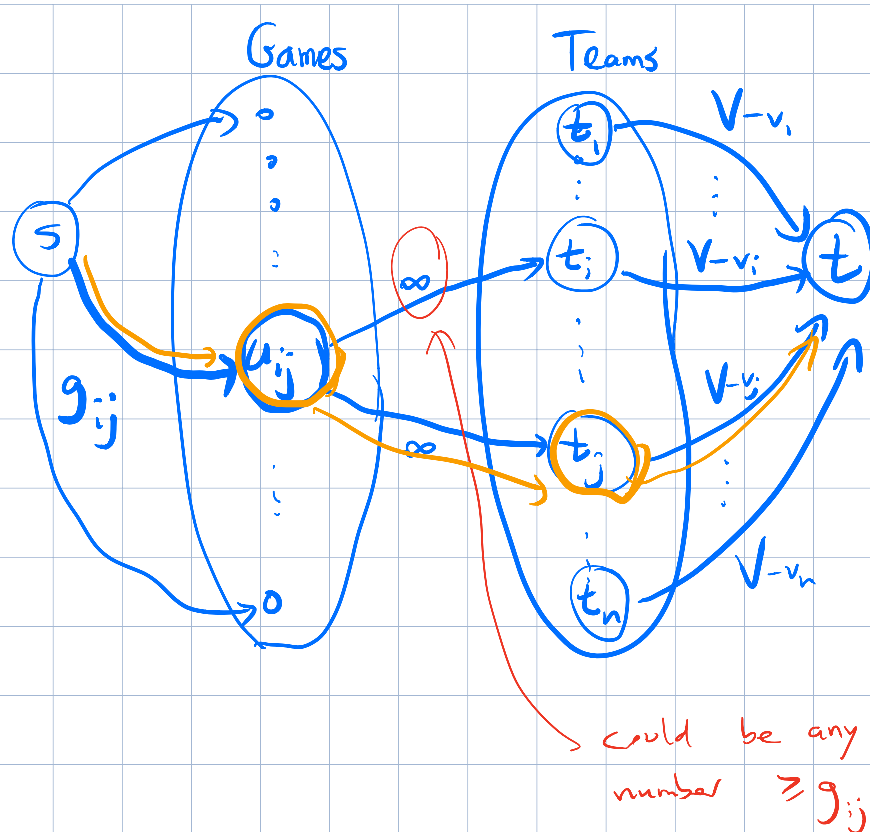
How to solve the general case?
Given a finite list of (team, # victories) pairs such as (NY, 92).
Given a list of games remaining to be played....
Can Boston finish at least tied for 1^{st}?

Say teams are $t_0 = \text{Boston}, t_1, \ldots, t_n$.
Say # victories $v_0 \quad v_1, \ldots, v_n$.
Say g_{ij} = # games remaining between t_i & t_j.

Then if Boston all remaining games it'll have

$$v_0 + g_{01} + g_{02} + \cdots + g_{0n} = V \text{ victories.}$$

We want to know if it's possible to select an outcome for all other games such that each t_i $(i > 0)$ gets at most $V - v_i$ additional victories.



Question. Is the max-flow value equal to sum of capacities of all edges leaving s?

If so, Boston can finish in first.
If not, then it's impossible.

Exercise. Use max-flow min-cut theorem to show that if Boston is mathematically eliminated there always a way to prove it by identifying a set of teams that collectively can have at most K victories but play $> K$ games against one another.

Circulations with Demands and Lower Bounds (§7.7)

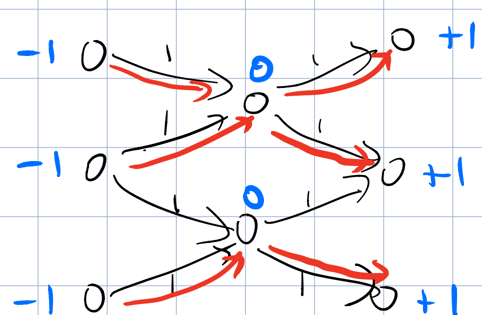
Def. Given directed $G = (V, E)$ with capacity function $c : E \to \mathbb{R}_+$ and demand function $d : V \to \mathbb{R}$

a circulation satisfying demand d is a function $f : E \to \mathbb{R}_{\geq 0}$ such that

- [capacity constr] $0 \leq f(e) \leq c(e) \quad \forall e$
- [demand constr]

$$\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v) \quad \forall v.$$

Ex.



Given a circulation with demands problem, does $\exists$ a circulation that satisfies the demands?

Reduce to max flow: build network $G^+ = (V^+, E^+)$, where

$$V^+ = V \cup \{s, t\}$$

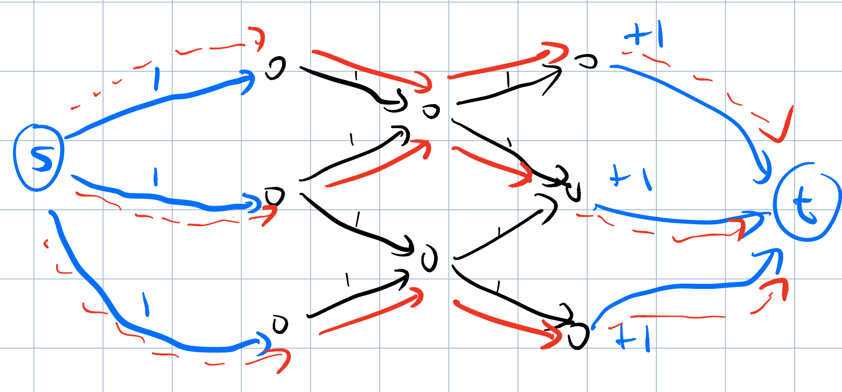
$$E^+ = E \cup E_s \cup E_t$$

edges leaving s

edges entering t

E_s contains an edge (s, u) with capacity $-d(u)$ for all $u \in V$ s.t. $d(u) < 0$.

E_t contains edge (v, t) with cap. $d(v)$ for all $v \in V$ s.t. $d(v) > 0$.



$\{$ circulations in G that satisfy demands $\}$

$$\updownarrow$$

$\{$ flows in G^+ that saturate all edges in E_s and E_t $\}$