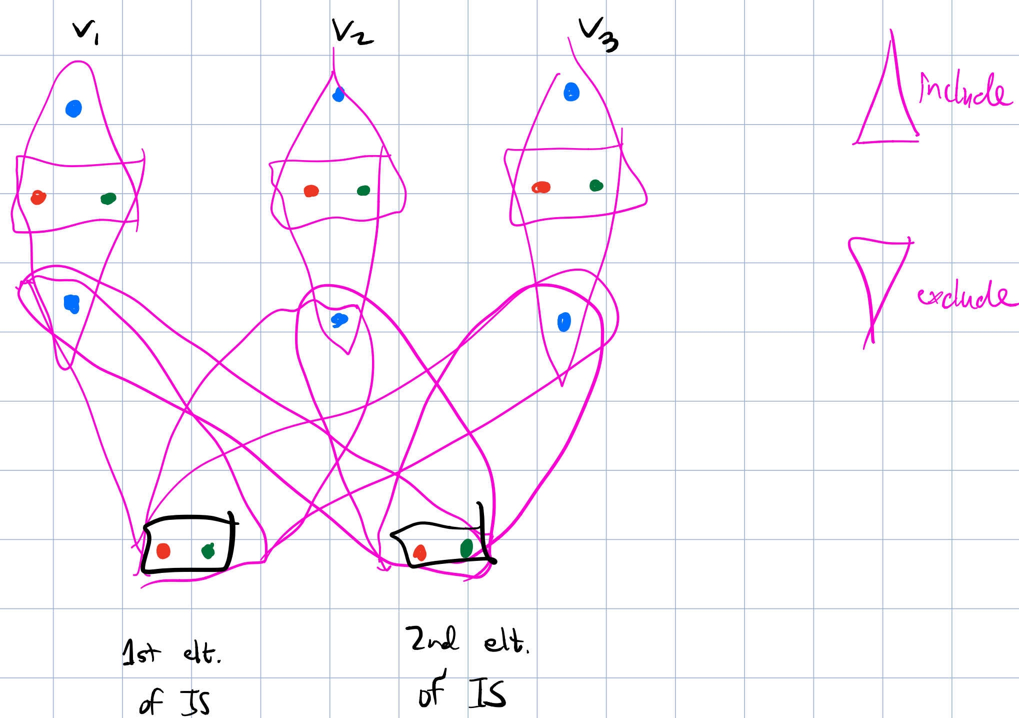
$\boxed{\text{9 April 2021}}$ Three-Dimensional Matching and Subset Sum

Announcement: Check pinned Piazza post for Prelim 2 review materials.

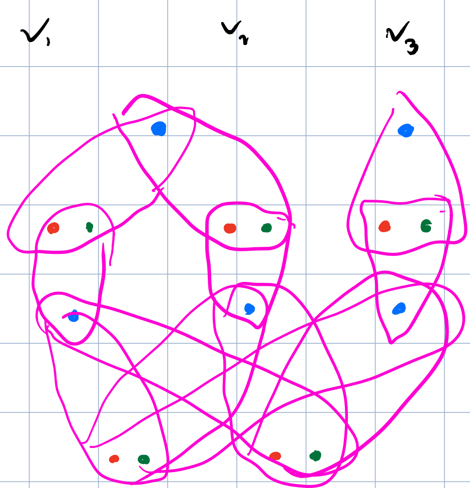
$\boxed{k=2:}$

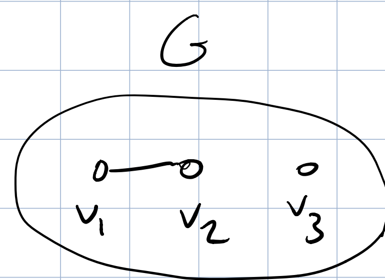


Recap: 3DM is to decide if the 3n elements can be partitioned into disjoint 3-element sets, each of which is among the given ones

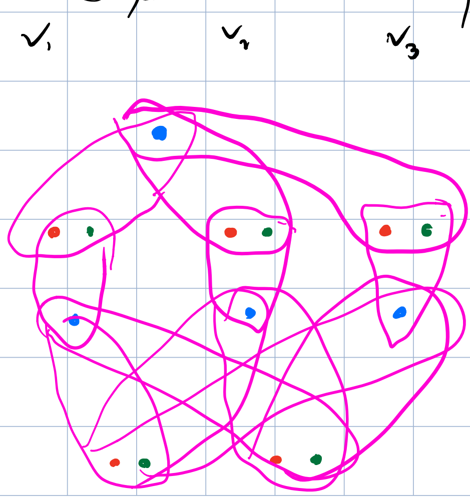
Q1. How to express the "don't pick both endpoints of an edge" constraint using 3DM?

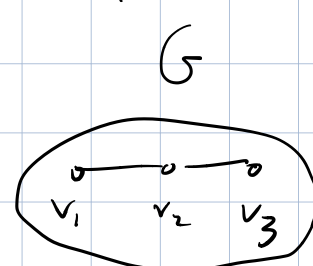
Q2. What to do with leftover blue nodes?





Trouble: if a vertex has > 1 neighbor, it becomes impossible to represent the edges of the graph accurately.





Solution: use a "pinwheel gadget" for each vertex.

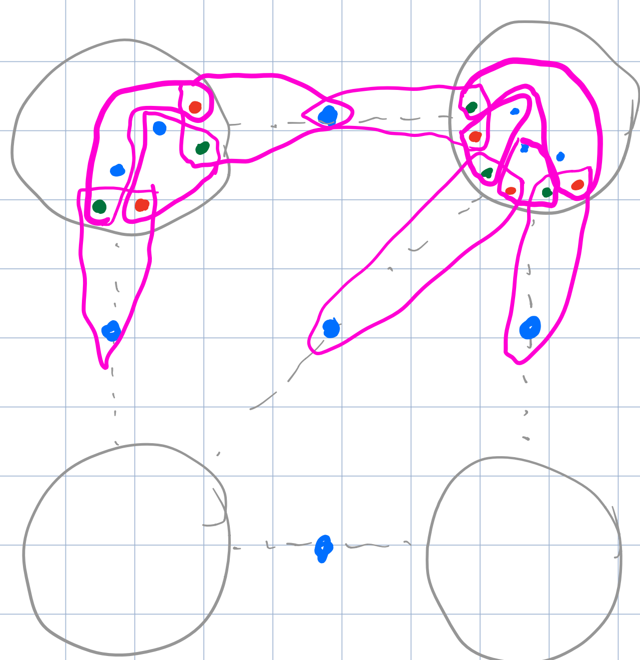
If the vertex belongs to d edges, the pinwheel will have

d red nodes

d green nodes

$2d$ blue nodes

(d of them correspond to "choosing" the vertex and will be shared with pinwheels of neighboring vertices.)



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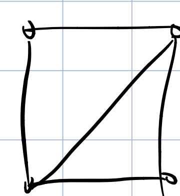

Do this for each vertex.

There are 2 ways to cover the red-green nodes in a vertex gadget.

1. Connect each red to clockwise green neighbor. Covers all the blue nodes on the edges leaving that vertex.

   "Choosing the vertex to belong to the indep set."

2. Connect to counterclockwise neighbor. Covers all the internal blue nodes.

Enforce "choose k vertices" constraint by having k additional red-green pairs each connected to one INTERNAL blue node of <u>each</u> pinwheel.

\# edges v belongs to

Total: $k + \sum_{v \in V} d(v)$ red

" " " green

\# edges in the ind set

external

$m + \sum_{v \in V} d(v)$ blue

internal.

graph

Property: There exists a 3D matching that covers all red & green nodes (not necessarily all blue)
$\Longleftrightarrow$ G has a k-element indep set.

Summary: We showed a problem is NP-Complete but it's not 3D matching. It is "3D Matching with leftover blue nodes" (3DMWLB)

Given an input instance of 3DM can we find disjoint 3-elements sets among the given ones that cover all red and green nodes?

Finally: show 3DMWLB $\leq_p$ 3DM.

Given input instance of 3DMWLB with sets R, G, B, $|R| = n$, $|G| = n$, $|B| = n+s$ $(s \geq 0)$
create a 3DM instance with sets R', G', B
$R' = R \cup R_1$ $|R_1| = s$
$G' = G \cup G_1$ $|G_1| = s$

and the 3-element sets in this 3DM instance are all the sets given in the original 3DMWLB instance, and $s^2 \cdot (n+s)$ additional sets: each set formed from one element of R_1, one elt of G_1, one element of B.

We've shown IND SET $\leq_p$ 3DMWLB $\leq_p$ 3DM.

<u>SUBSET SUM.</u> Given positive integers $w_1, w_2, \ldots, w_n$ and target sum W is there a subset of $\{w_1, \ldots, w_n\}$ that sums up to W?

<u>KNAPSACK.</u> Given items with integer weights and values (w_i, v_i), and budget B and target value V, is there a subset with combined weight $\leq B$ and combined value $\geq V$?

SUBSET SUM $\leq_p$ KNAPSACK

$w_i \longmapsto$ item with $v_i = w_i$

target $W \longmapsto B = V = W$.

Recall the dynamic program for knapsack runs in $O(nW)$ time, which is <u>pseudopolynomial</u> but not polynomial.

If W has m binary digits, then the knapsack problem has input size $O(nm)$ but the dyn prog takes $O(n \cdot 2^m)$ time to solve it.

Showing SUBSET SUM is NP-complete.

Reduce 3DM $\leq_p$ SUBSET SUM.