

Lecture 26

Topics

1. Brief review of BRFT with partial types (it is a constructive theory, call it CBRFT).
2. The Hatling Problem for middle school students, Theorem 3.3.
3. Classes, CT, and Rice's Theorem for partial types.
4. Other BRFT results, open problems, and general lessons (connections to Hoare logic).

Uniformity principles 4.1.

Dovetailing results (Why no universal machine? Church/Turing thesis).

Why not Church's Thesis? Free choice sequences, distributed computing.

For Lecture 27 on Friday March 27 we will examine the Blum size theorem which tells us about subrecursive languages vs. universal (partial-recursive) languages.

Theorem of $\overline{\text{LCF}}/\overline{\text{PCF}}/\overline{\text{CBRFT}}$ (Constructive Recursive Function Theory) *Given any non-empty type T , there is no function $h : \bar{T} \rightarrow \mathbb{B}$ such that $\forall x. \bar{T}(h(x) = tt \text{ iff } x \downarrow)$, e.g. no internal computable h solves the halting problem on \bar{T} .*

Consider the special case of $T = N$ for concreteness, but the proof works verbatim for any non-empty type T .

Corollary $\neg \exists h : \bar{\mathbb{N}} \rightarrow \mathbb{N}. \forall x : \bar{\mathbb{N}} (h(x) = 0 \text{ iff } x \downarrow)$

Suppose such an $h : \bar{\mathbb{N}} \rightarrow \mathbb{N}$ could be constructed in this theory, then let $d = \text{fix}(\lambda x. \text{zero}(h(x); \perp; 0))$ (note $d \in \bar{\mathbb{N}}$) for the zero function defined in Lecture 25 and the article on computational foundations of BRFT.

Let $f = \lambda x. \text{zero}(h(x); \perp; 0) : \bar{\mathbb{N}} \rightarrow \mathbb{N}$. So $\text{fix}(f) = d \in \bar{\mathbb{N}}$ by the fixed point rule.

Now consider the value of $h(d)$. The value belongs to \mathbb{N} , so we have two cases, $(h(d) = 0) \vee (h(d) \neq 0)$. We show that neither case is possible, hence we have no such h .

1. If $h(d) = 0$ then $zero(h(d); \perp; 0) = \perp$, contrary to the defining requirement on h .
2. If $h(d) \neq 0$, then $zero(h(d); \perp; 0) = 0$, by the definition the value should be \perp , a diverging element.

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Exercise. Prove that no internal function solves the divergence problem, i.e. the class $div K_T = \{x : \bar{T}|x \uparrow\}$ is not decidable.