predicate, then $\nu'(r) \simeq \nu'(s)$. We shall show that then ν' is not partial recursive. For each k, let partial recursive predicates R_k and S_k be defined thus:

$$R_k(y) \cong y=0 \vee y=1+0\cdot\mu z T_1(k, k, z), \quad S_k(y) \cong y=0\cdot\mu z T_1(k, k, z) \vee y=1.$$

Let r_k and s_k be any Gödel numbers of R_k and S_k, respectively. Given k, $R_k(0)$ and $S_k(1)$ are true; so by (a), $\nu'(r_k)$ and $\nu'(s_k)$ are both defined. If $(Ez)T_1(k, k, z)$, then $(y)[R_k(y) \cong S_k(y)]$, i.e. R_k and S_k are the same predicate; and hence using (b), $\nu'(r_k)=\nu'(s_k)$. Thus $(Ez)T_1(k, k, z) \rightarrow \nu'(r_k)=\nu'(s_k)$, or contraposing (cf. *12 § 26), $\nu'(r_k) \neq \nu'(s_k) \rightarrow \overline{(Ez)}T_1(k, k, z)$. Conversely, if $\overline{(Ez)}T_1(k, k, z)$, then $R_k(y)$ is true only for $y = 0$ and $S_k(y)$ only for $y = 1$, so by (a), $\nu'(r_k) \neq \nu'(s_k)$. Thus $\nu'(r_k) \neq \nu'(s_k) \equiv \overline{(Ez)}T_1(k, k, z) \equiv (z)\overline{T}_1(k, k, z)$. The expressions defining $R_k(y)$ and $S_k(y)$ are partial recursive predicates of the two variables k, y; say those predicates have Gödel numbers r and s, respectively. We can take $r_k = S^1_1(r, k)$ and $s_k = S^1_1(s, k)$ in the above, obtaining the equivalence $\nu'(S^1_1(r, k)) \neq \nu'(S^1_1(s, k)) \equiv (z)\overline{T}_1(k, k, z)$. If ν' were partial recursive, the left member would be a general recursive predicate of k; but the right is not (Theorem V (14)).

§ 66. The recursion theorem. THEOREM XXVI. *For any $n \geq 0$, let $\mathsf{F}(\zeta; x_1, \ldots, x_n)$ be a partial recursive functional, in which the function variable ζ ranges over partial functions of n variables. Then the equation*

$$\zeta(x_1, \ldots, x_n) \simeq \mathsf{F}(\zeta; x_1, \ldots, x_n)$$

has a solution φ for ζ such that any solution φ' for ζ is an extension of φ, and this solution φ is partial recursive.

Similarly, when Ψ are l partial functions and predicates,

$$\zeta(x_1, \ldots, x_n) \simeq \mathsf{F}(\zeta, \Psi; x_1, \ldots, x_n)$$

has a solution φ for ζ such that any solution φ' for ζ is an extension of φ, and this solution φ is partial recursive in Ψ. (The first recursion theorem.)

PROOF (for $l = 0$, $n = 1$). Let φ_0 be the completely undefined function. Then introduce $\varphi_1, \varphi_2, \varphi_3, \ldots$ successively by

$$\varphi_1(x) \simeq \mathsf{F}(\varphi_0; x), \quad \varphi_2(x) \simeq \mathsf{F}(\varphi_1; x), \quad \varphi_3(x) \simeq \mathsf{F}(\varphi_2; x), \ldots$$

Since φ_0 is completely undefined, φ_1 is an extension of φ_0; then by Theorem XXI (a), φ_2 is an extension of φ_1, φ_3 of φ_2, etc. Let φ be the "limit function" of $\varphi_0, \varphi_1, \varphi_2, \ldots$; i.e. for each x, let $\varphi(x)$ be defined if and only if $\varphi_s(x)$ is defined for some s, in which case its value is the common value of $\varphi_s(x)$ for all $s \geq$ the least such s. Now:

(i) *For each x, $\varphi(x) \simeq \mathsf{F}(\varphi; x)$.* For consider any x. Suppose $\varphi(x)$ is defined. Then for some s, $\varphi(x) \simeq \varphi_{s+1}(x)$ [by definition of φ] $\simeq \mathsf{F}(\varphi_s; x)$ [by definition of φ_{s+1}] $\simeq \mathsf{F}(\varphi; x)$ [by Theorem XXI (a), since φ is an extension of φ_s]. Conversely, suppose $\mathsf{F}(\varphi; x)$ is defined; call its value k. Since F is partial recursive, there is a system F of equations defining $\mathsf{F}(\zeta; x)$ recursively from ζ, say with f as principal and g as given function letter; so now there is a deduction of $\mathrm{f}(\boldsymbol{x})=\boldsymbol{k}$ from $\mathrm{E}^{\varphi}_{\mathrm{g}}$, F. Let $\mathrm{g}(\boldsymbol{y}_1)=\boldsymbol{z}_1, \ldots, \mathrm{g}(\boldsymbol{y}_p)=\boldsymbol{z}_p$ (where $z_i = \varphi(y_i)$) be the equations of $\mathrm{E}^{\varphi}_{\mathrm{g}}$ occurring in this deduction. But $\varphi(y_1) = \varphi_{s_1}(y_1), \ldots, \varphi(y_p) = \varphi_{s_p}(y_p)$ for some $s_1, \ldots, s_p$. Let $s = \max(s_1, \ldots, s_p)$. Then $\varphi(y_1) = \varphi_s(y_1), \ldots, \varphi(y_p) = \varphi_s(y_p)$. So $\mathrm{g}(\boldsymbol{y}_1)=\boldsymbol{z}_1, \ldots, \mathrm{g}(\boldsymbol{y}_p)=\boldsymbol{z}_p \in \mathrm{E}^{\varphi_s}_{\mathrm{g}}$. Thus $\mathrm{E}^{\varphi_s}_{\mathrm{g}}, \mathrm{F} \vdash \mathrm{f}(\boldsymbol{x})=\boldsymbol{k}$. Hence $k \simeq \mathsf{F}(\varphi_s; x) \simeq \varphi_{s+1}(x) \simeq \varphi(x)$.

(ii) *If for each x, $\varphi'(x) \simeq \mathsf{F}(\varphi'; x)$, then φ' is an extension of φ.* It will suffice to show by induction on s that, for each x, if $\varphi_s(x)$ is defined, then $\varphi'(x) \simeq \varphi_s(x)$. BASIS: $s = 0$. True vacuously. IND. STEP. Suppose for a given x that $\varphi_{s+1}(x)$ is defined. Then $\varphi_{s+1}(x) \simeq \mathsf{F}(\varphi_s; x) \simeq \mathsf{F}(\varphi'; x)$ [by Theorem XXI (a), since by hyp. ind. φ' is an extension of φ_s] $\simeq \varphi'(x)$.

(iii) *If* F *defines $\mathsf{F}(\zeta; x)$ recursively from ζ, and* E *comes from* F *by substituting the principal function symbol* f *for the given function symbol* g, *then* E *defines φ recursively.* It will suffice to show that $\mathrm{E} \vdash \mathrm{f}(\boldsymbol{x})=\boldsymbol{k}$, if and only if $\varphi_s(x) \simeq k$ for some s. We easily see that if $\varphi_s(x) = k$, then $\mathrm{E} \vdash \mathrm{f}(\boldsymbol{x})=\boldsymbol{k}$. For the converse, we show by induction on h that if there is a deduction of $\mathrm{f}(\boldsymbol{x})=\boldsymbol{k}$ from E of height h, then $\varphi_s(x) = k$ for some s. The deduction can be altered if necessary, so that in each inference by R2 with a minor premise of the form $\mathrm{f}(\boldsymbol{y})=\boldsymbol{z}$ only one occurrence of $\mathrm{f}(\boldsymbol{y})$ in the major premise is replaced by $\boldsymbol{z}$ (ACT 1). The occurrences of f in equations of the deduction can be classified in an evident manner into those which come from an occurrence of f in F, and those which come via the substitution of f for g from an occurrence of g in F. Now consider the inferences by R2 with minor premise of the form $\mathrm{f}(\boldsymbol{y})=\boldsymbol{z}$ in which the f of the part $\mathrm{f}(\boldsymbol{y})$ replaced comes from a g in F. Say there are p such inferences, the minor premises $\mathrm{f}(\boldsymbol{y}_1)=\boldsymbol{z}_1, \ldots, \mathrm{f}(\boldsymbol{y}_p)=\boldsymbol{z}_p$ of which do not stand above other such premises. Each of these p premises occurred above the endequation of the given deduction before Act 1; so using the hypothesis of the induction, $z_1 \simeq \varphi_{s_1}(y_1) \simeq \varphi_s(y_1), \ldots, z_p \simeq \varphi_{s_p}(y_p) \simeq \varphi_s(y_p)$ where $s = \max(s_1, \ldots, s_p)$. Now consider the tree remaining from the deduction after Act 1, when all the equations above $\mathrm{f}(\boldsymbol{y}_1)=\boldsymbol{z}_1, \ldots, \mathrm{f}(\boldsymbol{y}_p)=\boldsymbol{z}_p$ are removed (ACT 2). In this tree, let each occurrence of f which (before Act 2) came from a g of F be changed back to g (ACT 3). The f's in question all occurred in the right members of equations, since g being the given function symbol of F occurs in F only on the right; so no f is changed by Act 3 in what was a minor premise for R2 before Act 3 or in the end-equation $f(\boldsymbol{x})=\boldsymbol{k}$. Finally, let the f's of $f(\boldsymbol{y}_1)=\boldsymbol{z}_1, \ldots, f(\boldsymbol{y}_p)=\boldsymbol{z}_p$ be changed to g (Act 4), which restores the inferences by R2 which Act 3 spoiled. The resulting tree is a deduction of $f(\boldsymbol{x})=\boldsymbol{k}$ from $E_g^{\varphi_s}$, F. Hence

$$k \simeq F(\varphi_s; x) \simeq \varphi_{s+1}(x).$$

Example 1. Consider the problem: to find a partial recursive function φ such that

(a) $$\varphi(x) \simeq \varphi(x);$$

i.e. to solve the equation $\zeta(x) \simeq \zeta(x)$ for ζ. Obviously any partial function satisfies this equation. The partial function with the least range of definition which satisfies is the completely undefined function. This is the solution φ given by the theorem (with $F(\zeta;\, x) \simeq \zeta(x)$).

Example 2. To find a partial recursive function φ such that

(a) $$\varphi(x) \simeq \varphi(x)+1;$$

i.e. to solve $\zeta(x) \simeq \zeta(x)+1$ for ζ. Only the completely undefined partial function satisfies. This of course is the solution φ given by the theorem (with $F(\zeta; x) \simeq \zeta(x)+1$).

Example 3. To find a function φ partial recursive in χ such that

(a) $$\begin{cases} \varphi(0) \simeq q, \\ \varphi(y') \simeq \chi(y, \varphi(y)) \end{cases}$$

(Schema (Va) § 43). Only one function φ satisfies for a given χ, and we already know by Theorem XVII (a) that it is partial recursive in χ. However to see how the theorem applies, we rewrite (a) as

(b) $$\varphi(x) \simeq F(\varphi, \chi; x) \text{ where } F(\zeta, \chi; x) \simeq \begin{cases} q & \text{if } x=0, \\ \chi(x \dot{-} 1, \zeta(x \dot{-} 1)) & \text{if } x>0 \end{cases}$$

(equivalently,

$$F(\zeta, \chi; x) \simeq \mu w[\{x=0 \;\&\; w=q\} \vee \{x>0 \;\&\; w=\chi(x \dot{-} 1, \zeta(x \dot{-} 1))\}]).$$

Since $F(\zeta, \chi; x)$ is partial recursive (using Theorems XVII, XX (c); or XVII, XVIII, XX (a)), by the theorem φ is partial recursive in χ.

Example 4. We give a new proof of Theorem XVIII (which proof in various guises appeared in Kleene 1935, 1936, 1943). Let $\varphi(x) \sim \mu y[\chi(x, y)=0]$. Then $\varphi(x) \simeq \varphi(x, 0)$ where

(a) $$\varphi(x, y) \simeq \mu t_{t \geq y}[\chi(x; t)=0].$$

But $\varphi(x, y)$ is the partial function φ with the least range of definition such that

(b) $$\varphi(x, y) \simeq \begin{cases} y & \text{if } \chi(x, y)=0, \\ \varphi(x, y') & \text{if } \chi(x, y) \neq 0. \end{cases}$$

By Theorem XX (c) (with the first proof), the right side of (b) is of the form $F(\varphi, \chi; x, y)$ where $F(\zeta, \chi; x, y)$ is partial recursive; so by the present theorem $\varphi(x, y)$, and hence $\varphi(x)$, is partial recursive in χ.

Discussion. The theorem for $l=0$ asserts that we can impose any relationship of the form

(72) $$\varphi(x_1, \ldots, x_n) \simeq F(\varphi; x_1, \ldots, x_n)$$

expressing the ambiguous value $\varphi(x_1, \ldots, x_n)$ of a function φ in terms of φ itself and $x_1, \ldots, x_n$ by methods already treated in the theory of partial recursive functions; and conclude that the partial function with the least range of definition which satisfies the relationship is partial recursive.

Moreover the case of the theorem for $l>0$, in which

(73) $$\varphi(x_1, \ldots, x_n) \simeq F(\varphi, \Psi; x_1, \ldots, x_n)$$

is the relationship imposed, can be used to extend the body of the methods available for use in further applications.

In our examples of special kinds of "recursion" (§§ 43, 46 and beginning § 55) the ambiguous function value $\varphi(x_1, \ldots, x_n)$ was expressed in terms of values of the same function for sets of arguments preceding the given n-tuple $x_1, \ldots, x_n$ in terms of some special ordering of the n-tuples. We now have a general kind of "recursion", in which the value $\varphi(x_1, \ldots, x_n)$ can be expressed as depending on other values of the same function in a quite arbitrary manner, provided only that the rule of dependence is describable by previously treated effective methods.

The given "recursion" may now be ambiguous as a definition of an ordinary (i.e. completely defined) number-theoretic function φ, in the sense that it is satisfied by more than one such function (Example 1, or (b) in Example 4 when $\chi(x, y)$ does not vanish for infinitely many values of y). But now we choose as the solution which interests us that partial function which is defined only when the recursion requires it to be. The given "recursion" may be inconsistent as a definition of an ordinary function (Example 2); again the difficulty is escaped now through the fact that it is only a partial function which we are seeking as the solution. Both these situations can arise when the F is general recursive (Examples 1 and 2). When F is incompletely defined, the recursion may also directly