uniform in f, g such that

   3. $\varphi_i = \varphi_{g(j)}$

   4. $f \mid i \mid \, < \mid g(j) \mid$ *(i.e., M_i is considerably smaller than $M_{g(j)}$).*

EXAMPLE. If $f(x) = 100\,x$, then $M_{g(j)}$ is 100 times as large as M_i, though both $M_{g(j)}$ and M_i compute the same function.

Proof. We give a procedure for determining the two integers i and j uniformly in f, g. Consider the following set of instructions:

   "With inputs x and y, first compute $f \mid y \mid$. Then compute $g(0)$, $g(1)$, $\cdots$ until the least j is found such that $f \mid y \mid \, < \mid g(j) \mid$. Then compute $\varphi_{g(j)}(x)$ and give $\varphi_{g(j)}(x)$ as output."

These instructions define a partial recursive function $\varphi_z^2(x, y)$ whose index z is uniform in f, g. The recursion theorem[1] then supplies an integer i which is uniform in f, g such that

$$Equation\ 1\colon\ \varphi_i(x) = \varphi_z^2(x, i)\ \text{for all } x.$$

We shall show that this is the desired i.

Conditions 1, 2 ensure that we can find a j uniformly in f, g that satisfies 4, $f \mid i \mid \, < \mid g(j) \mid$: First determine i, which we have shown to be uniform in f, g. Then compare $f \mid i \mid$ with $\mid g(0) \mid$, $\mid g(1) \mid$, $\cdots$ until a j is found that satisfies 4. For this j (cf. Eq. 1 and above instructions) condition 3 is satisfied, $\varphi_i(x) = \varphi_z^2(x, i) = \varphi_{g(j)}(x)$ for all x.   Q.E.D.

g is an algorithmic function for enumerating an infinite set of machines. The constructive nature of this proof enables one to effectively replace g by a function g' which enumerates machines that are no larger, and sometimes are considerably smaller than those enumerated by g.

It has been said that since practically all computable functions are primitive recursive, one does not need general recursion for any *practical* purposes. Theorem 1, though, gives practical reasons for favoring general recursion: It implies that there exists a primitive recursive function whose smallest derivation (defining equations) in the primitive recursive format is considerably larger than its smallest derivation in the general recursive format. More precisely, suppose primitive and general recursion are defined by derivations as in Davis (1958). Take the size of a derivation to be the number of letters in it. Then each primitive recursive function has at least one *smallest* primitive recursive derivation. The set of all such smallest derivations is r.e.; let g enumerate them. Upon setting $f(x) = n \cdot x$, the theorem supplies a primitive recursive

[1] The recursion theorem asserts that for every partial recursive function h there exists an integer i which is uniform in h such that $\varphi_i(x) = h(x, i)$ for all x.