Assignment 12

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Definition. The multicategory **LMet** has Lawvere metric spaces as its objects and n-ary metric maps as its multimorphisms, where an n-ary metric map from $[\langle X_1, d_1\rangle, \ldots, \langle X_n, d_n\rangle]$ to $\langle Y, d\rangle$ is an n-ary function $f : [X_1, \ldots, X_n] \to Y$ with the following property:

$$\forall x_1, x_1' \in X_1, \ldots, x_n, x_n' \in X_n.\ \ d_1(x_1, x_1') + \cdots + d_n(x_n, x_n') \geq d(f(x_1, \ldots, x_n), f(x_1', \ldots, x_n'))$$

Identities and compositions of n-ary metric maps are given by identities and compositions of the underlying n-ary functions.

Exercise 1. Give a direct definition of an **LMet**-enriched category, i.e. a definition that someone starting this class would be able to understand (even though they would not have an intuition for what it means). You may assume the definitions of a category and of arithmetic on the nonnegative extended reals (but not of a Lawvere metric space) are already understood. Do not give the proof that your definition is in fact equivalent to a **LMet**-enriched category.

Exercise 2. Show that one can make the category **LMet** into an **LMet**-enriched category.

Remark. Interestingly, one can alternatively view **LMet** as a part of an $\mathbf{R}^{>}_{*\leq}$-*classified* category, where $\mathbf{R}^{>}_{*\leq}$ is the multiorder on the positive reals given by $r_1 * \cdots * r_n \leq r'$. An r-classified morphism $f : \langle X, d_X\rangle \xrightarrow{r} \langle Y, d_Y\rangle$ of this $\mathbf{R}^{>}_{*\leq}$-classified category is a function $f : X \to Y$ satisfying the following property:

$$\forall x \in X.\ \ r * d_X(x, x') \geq d_Y(f(x), f(x)')$$

LMet, then, is the category of 1-classified morphisms, which works out to form a category because 1 is an internal monoid of $\mathbf{R}^{>}_{*\leq}$.

Exercise 3. Suppose a category $\mathbf{C}$ has binary products, and suppose I is an object of $\mathbf{C}$. The simple-slice category over I, denoted $\mathbf{C}//I$, has the same objects as $\mathbf{C}$, but a morphism of $\mathbf{C}//I$ from A to B is a morphism of $\mathbf{C}$ from $I \times A$ to B. Show that this construction extends to a $\mathbf{C}$-indexed category (which includes showing that $\mathbf{C}//I$ is a category).