2-Categories

Ross Tate

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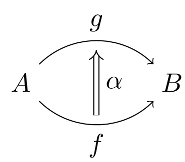
Definition (2-Category)**.** A (strict) 2-category is comprised of the following:

0-Cells (Objects) A set Ob of "0-cells", also known as objects.

1-Cells (Morphisms) For each pair of 0-cells A and B in Ob, a set $\mathrm{Hom}(A, B)$ of "1-cells from A to B", also known as morphisms. A 1-cell is often declared textually as $f : A \to B$ or graphically as $A \xrightarrow{f} B$.

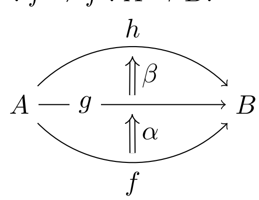
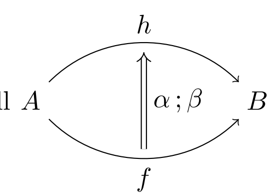
2-Cells For each pair of 0-cells A and B in Ob and each pair of 1-cells f and g in $\mathrm{Hom}(A, B)$, a set $\mathrm{Face}(f, g)$ of "2-cells from f to g". A 2-cell is often declared textually as $\alpha : f \Rightarrow g : A \to B$ or graphically as follows:

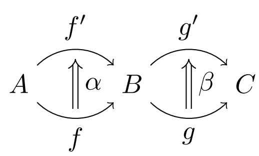
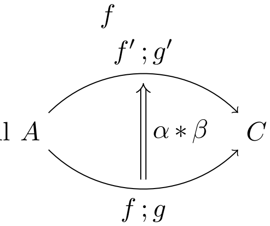


1-Identities For each 0-cell A, a 1-cell $id_A : A \to A$.

1-Composition For each chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, a 1-cell $A \xrightarrow{f\,;\,g} C$.

2-Identities For each 1-cell $f : A \to B$, a 2-cell $id_f : f \Rightarrow f : A \to B$.

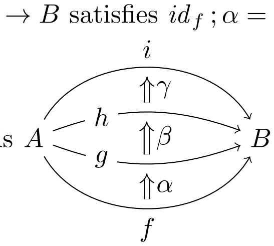
Vertical 2-Composition For each chain of 2-cells  , a 2-cell  .

Horizontal 2-Composition For each chain of 2-cells  , a 2-cell  .

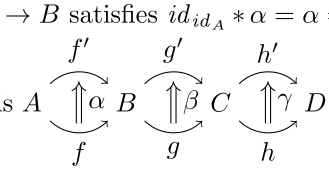
1-Identity Every 1-cell $f : A \to B$ satisfies $id_A\,;f = f = f\,;id_B$.

1-Associativity Every chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ satisfies $(f\,;g)\,;h = f\,;(g\,;h)$.

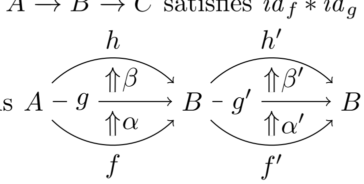
Vertical 2-Identity Every 2-cell $\alpha : f \Rightarrow g : A \to B$ satisfies $id_f\,;\alpha = \alpha = \alpha\,;id_g$.

Vertical 2-Associativity Every chain of 2-cells  satisfies $(\alpha\,;\beta)\,;\gamma = \alpha\,;(\beta\,;\gamma)$.

Horizontal 2-Identity Every 2-cell $\alpha : f \Rightarrow g : A \to B$ satisfies $id_{id_A} * \alpha = \alpha = \alpha * id_{id_B}$.

Horizontal 2-Associativity Every chain of 2-cells  satisfies $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

2-Identity Every sequence of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ satisfies $id_f * id_g = id_{f\,;\,g}$.

2-Interchange Every clover of 2-cells  satisfies $(\alpha\,;\beta) * (\alpha'\,;\beta') = (\alpha * \alpha')\,;(\beta * \beta')$.

Example. **Cat** is the 2-category of categories (as 0-cells), functors (as 1-cells), and natural transformations (as 2-cells).

Definition (2-Thin)**.** A 2-category is 2-thin if there is at most one 2-cell between any two given 1-cells. Consequently, when defining 2-thin 2-categories, one need only specify when one 1-cell is "less than" another 1-cell, indicating that there is a unique morphism from the former to the latter.

Example. Given a category $\mathbf{X}$, $\mathbf{Con}(\mathbf{X})$ is the 2-category of concrete categories over $\mathbf{X}$, concrete functors over $\mathbf{X}$, and identity-carried natural transformations over $\mathbf{X}$. **Con** is the 2-category $\mathbf{Con}(\mathbf{Set})$ of constructs. (Note that the textbook refers to these as $\mathbf{CAT}(\mathbf{X})$ and **CONST**.) Because the underlying functor of a concrete category is required to be faithful, one can prove that $\mathbf{Con}(\mathbf{X})$ is always 2-thin.

Example. **Rel** is the 2-thin 2-category obtained from the category **Rel** by defining $R \leq S : A \to B$ as

$$\forall a \in A, b \in B.\ a\ R\ b \implies a\ S\ b$$

Example. **Prost** is the 2-thin 2-category obtained from the category **Prost** by defining $f \leq g : \langle A, \leq \rangle \to \langle B, \leq \rangle$ as

$$\forall a \in A.\ f(a) \leq g(a)$$

Example. **LMet** is the 2-thin 2-category obtained from the category **LMet** by defining $f \leq g : \langle A, d \rangle \to \langle B, d \rangle$ as

$$\forall a, a' \in A.\ d(a, a') \geq d(f(a), g(a'))$$

Note that if $\langle B, d \rangle$ is separated, then $f \leq g$ implies $f = g$ since $\forall a \in A.\ 0 \geq d(a, a) \geq d(f(a), g(a)) \implies f(a) = g(a)$.

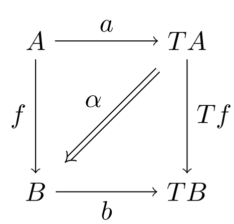
Definition. Given a 2-category $\mathbf{C}$, the 2-category $\mathbf{C}^{\mathrm{op}}$ is defined to have the same components but with the 1-cells reversed. Similarly, the 2-category $\mathbf{C}^{\mathrm{co}}$ is defined to have the same components but with the 2-cells reversed. Lastly, the 2-category $\mathbf{C}^{\mathrm{coop}}$ is defined to have the same components but with both the 1-cells and the 2-cells reversed. Note that $\mathbf{C}^{\mathrm{coop}}$, $(\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$, and $(\mathbf{C}^{\mathrm{op}})^{\mathrm{co}}$ are all the same.

Definition (2-Functor)**.** A 2-functor from a 2-category $\mathbf{C}$ to a 2-category $\mathbf{D}$ is a mapping of 0-cells, 1-cells, and 2-cells that preserves identities and compositions.

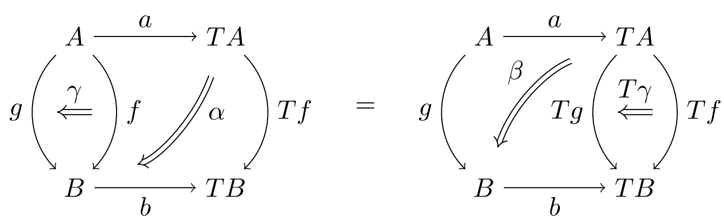
Definition. Given a 2-endofunctor $T : \mathbf{C} \to \mathbf{C}$, the 2-category $\mathbf{Coalg}_{\mathrm{colax}}(T)$ of coalgebras of T and colax algebra morphisms is comprised of the following:

Object An object A of $\mathbf{C}$ and a morphism $a : A \to TA$.

Morphism from $\langle A, a \rangle$ to $\langle B, b \rangle$ A morphism $f : A \to B$ and a 2-cell $\alpha : a\,;f \Rightarrow Tf\,;b : A \to TB$. That is:



2-Cell from $\langle f, \alpha \rangle$ to $\langle g, \beta \rangle$ A 2-cell $\gamma : f \Rightarrow g : A \to B$ such that the following 2-cells are equal:



Example. $\mathbf{Rel}(2)$ is isomorphic to the full subcategory of $\mathbf{Coalg}_{\mathrm{colax}}(\mathbb{P} : \mathbf{Prost} \to \mathbf{Prost})$ restricted to the objects $\langle\langle A, \leq \rangle, a \rangle$ for which $\leq$ is actually $=_A$. That is, $\mathbf{Rel}(2)$ is isomorphic to the category of $\mathbb{P}$-coalgebras on *sets* and colax morphisms of coalgebras. In theory this makes $\mathbf{Rel}(2)$ a 2-category, but one can prove that it is 2-discrete, meaning the only 2-cells are identities, and so it has no interesting 2-categorical structure.