Enriched Categories

Ross Tate

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Definition (Enriched Category over a Multicategory)**.** For a multicategory $\mathbf{M}$, an $\mathbf{M}$-enriched category is made of:

Objects A set Ob of objects

Morphisms For each A and B in Ob, an object $\mathrm{Hom}(A, B)$ of $\mathbf{M}$

Identities For each A in Ob, a nullary multimorphism $\mathsf{id}_A : [\,] \to_{\mathbf{M}} \mathrm{Hom}(A, A)$

Compositions For each A, B, and C in Ob, a multimorphism $\mathsf{comp}_{A,B,C} : [\mathrm{Hom}(A, B), \mathrm{Hom}(B, C)] \to_{\mathbf{M}} \mathrm{Hom}(A, C)$

Identity With the property that $\triangle_{\mathsf{id}_A, id_{\mathrm{Hom}(A,B)}} \mathsf{comp}_{A,B,C}$ and $\triangle_{id_{\mathrm{Hom}(A,B)}, \mathsf{id}_B} \mathsf{comp}_{A,B,C}$ equal $id_{\mathrm{Hom}(A,B)}$

Associativity And the property that $\triangle_{\mathsf{comp}_{A,B,C}, id_{\mathrm{Hom}(C,D)}} \mathsf{comp}_{A,C,D}$ equals $\triangle_{id_{Hom(A,B)}, \mathsf{comp}_{B,C,D}} \mathsf{comp}_{A,B,D}$

Example. A **Set**-enriched category is simply a category. Ob is the set of objects of the category, and $\mathrm{Hom}(A, B)$ is the set of morphisms from A to B. id_A is the identity morphism id_A, and $\mathsf{comp}_{A,B,C}$ is the composition function mapping $f : A \to B$ and $g : B \to C$ to $f \,;g : A \to C$. The identity and associativity requirements of **Set**-enriched categories are exactly the identity and associativity requirements of categories.

Example. Many examples of enriched categories are enriched over a multicategory $\mathbf{M}$ with an obvious faithful functor to **Set**. In such cases, $\mathbf{M}$-enriched categories can be viewed as categories whose sets of morphisms are enriched with additional $\mathbf{M}$-structure, and whose identities and compositions respect that structure. For example, a **Prost**-enriched category is a category along with a preorder on each set $\mathrm{Hom}(A, B)$ with the property that $f \le f'$ and $g \le g'$ implies $f \,; g \le f' \,; g'$. Thus a **Prost**-enriched category is a 2-thin 2-category. In fact, a 2-category is simply a **Cat**-enriched category: a 2-category is a category along with a category of 2-cells on each set $\mathrm{Hom}(A, B)$ of 1-cells, and with a notion of horizontal composition between connected hom-sets that respects the (vertical) categorical structure on these hom-sets. More generally, if we define $\mathbf{Cat}_0$ as an $\mathbf{1}$-enriched category (i.e. simply a set), and $\mathbf{Cat}_{n+1}$ as a $\mathbf{Cat}_n$-enriched category, then we arrive at the standard definitions of sets, categories, 2-categories, 3-categories, and so on. Interestingly, if we change the base case of the above series to $\mathbb{B}$-enriched categories, where $\mathbb{B}$ is the multiorder on the booleans given by $b_1 \wedge \cdots \wedge b_n \implies b'$, then we arrive at the sequence of preordered sets, preordered categories, preordered 2-categories, and so on.

Definition (Abelian Group)**.** An abelian group, i.e. a commutative group, is a group $\langle A, 0, +, - \rangle$ with the property that $a + b$ equals $b + a$ for all a and b in A. **Ab** is the multicategory of abelian groups and multilinear functions.

Definition (Preadditive Category)**.** A preadditive category is an **Ab**-enriched category. That is, it is a category along with an abelian group structure $\langle \mathrm{Hom}(A, B), 0, +, - \rangle$ on each hom-set such that the following properties hold:

$$0 \,; f = 0 = f \,; 0 \qquad (f + f') \,; g = (f \,; g) + (f' \,; g) \qquad f \,;(g + g') = (f \,; g) + (f \,; g')$$

Example. **Ab** is the **Ab**-enriched category in which, for abelian groups $\langle A, 0, +, - \rangle$ and $\langle B, 0, +, - \rangle$, the group structure on the set of group homomorphisms from $\langle A, 0, +, - \rangle$ to $\langle B, 0, +, - \rangle$ is given by defining 0 as $\lambda a \in A.\, 0$, by defining $f + g$ as $\lambda a \in A.\, f(a) + g(a)$, and by defining $-f$ as $\lambda a \in A.\, -f(a)$.

Theorem (Finite Biproducts)**.** *An object is a product of a list of objects $A_1, \ldots, A_n$ if and only if it is their coproduct.*

Proof. We prove only one direction since the other follows the exact some reasoning dualized. Suppose $\{P \xrightarrow{\pi_i} A_i\}_{i \in \{1,\ldots,n\}}$ is a product. Then define $\kappa_i : A_i \to P$ as the unique morphism with the property that $\kappa_i \,; \pi_j$ equals 0 when $i \ne j$ and id_{A_i} when $i = j$. Given a sink $\{f_i : A_i \to B\}_{i \in \{1,\ldots,n\}}$, define $[f_i]_{i \in \{1\ldots n\}} : P \to B$ as $\sum_{i \in \{1,\ldots,n\}} \pi_i \,; f_i$. This satisfies the required property as shown below:

$$\forall j.\ \kappa_j \,;[f_i]_{i \in \{1\ldots n\}} = \kappa_j \,; \textstyle\sum_i \pi_i \,; f_i = \sum_i \kappa_j \,; \pi_i \,; f_i = (id_{A_i} \,; f_i) + \left(\sum_{i | i \ne j} 0 \,; f_i\right) = f_i + \sum_{i|i \ne j} 0 = f_i + 0 = f_i$$

$$\left(\forall j.\ (\textstyle\sum_i \pi_i \,; \kappa_i) \,; \pi_j = \sum_i \pi_i \,; \kappa_i \,; \pi_j = (\pi_j \,; id_{A_j}) + \left(\sum_{i|i \ne j} \pi_i \,; 0\right) = \pi_j + 0 = \pi_j = id_P \,; \pi_j\right) \implies (\textstyle\sum_i \pi_i \,; \kappa_i) = id_P$$

$$\forall g : P \to B.\ (\forall i.\ \kappa_i \,; g = f_i) \implies g = id_P \,; g = (\textstyle\sum_i \pi_i \,; \kappa_i) \,; g = \sum_i \pi_i \,; \kappa_i \,; g = \sum_i \pi_i \,; f_i = [f_i]_{i \in \{1\ldots n\}}$$

$\square$

Remark. Note that the above proof actually applies to any **CommMon**-enriched category.

Definition (Additive Category)**.** An additive category is a preadditive category with all finite (co/bi)products.

Example. **Ab** is also an additive category. The zero group is the singleton group. The biproduct of $\langle A, 0, +, - \rangle$ and $\langle B, 0, +, - \rangle$ is the set $A \times B$ with identity $\langle 0, 0 \rangle$, with $\langle a, b \rangle + \langle a', b' \rangle = \langle a + a', b + b' \rangle$, and with $-\langle a, b \rangle = \langle -a, -b \rangle$.

Example. Let $\mathbf{R}^{\geq\infty}_{+\geq}$ be the multiorder on the nonnegative extended reals $\mathbb{R}^{\geq\infty}$ given by $r_1 + \cdots + r_n \geq r'$. An $\mathbf{R}^{\geq\infty}_{+\geq}$-enriched category is a Lawvere metric space. The objects are the points of the space. The $\mathbf{R}^{\geq\infty}_{+\geq}$-object $\mathrm{Hom}(A, B)$ is a nonnegative extended real number representing the distance from A to B. The nullary multimorphism $\mathsf{id}_A : [\,] \to \mathrm{Hom}(A, A)$ is a proof that 0 is greater than or equal to the distance from A to itself. The binary multimorphism $\mathsf{comp}_{A,B,C} : [\mathrm{Hom}(A, B), \mathrm{Hom}(B, C)] \to \mathrm{Hom}(A, C)$ is a proof that the distance from A to B plus the distance from B to C is greater than or equal to the distance from A to C. The identity and associativity requirements hold trivially due to thinness of $\mathbf{R}^{\geq\infty}_{+\geq}$.

Definition (Enriched Category over a Multicategory)**.** Given a colax/weak/strict monoidal category $\mathbf{M}$ there is a corresponding multicategory with the same objects and with multimorphisms from $[A_1, \ldots, A_n]$ to B being morphisms of $\mathbf{M}$ from $A_1 \otimes \cdots \otimes A_n$ to B. A category enriched over the monoidal category $\mathbf{M}$ is a category enriched over the multicategory corresponding to $\mathbf{M}$.

Definition (Suspension)**.** Given a multicategory $\mathbf{M}$ and a set $\mathcal{C}_0$ "of 0-cells", the Path-multicategory $\mathbf{Susp}_{\mathcal{C}_0}(\mathbf{M})$ is comprised of the following:

0-Cells A 0-cell is an element of $\mathcal{C}_0$

Vertical 1-Cells, Identities, and Compositions There is one vertical cell for each 0-cell, which is in turn the identity on that 0-cell, and which makes composition trivial

Horizontal 1-Cells For every pair of 0-cells A and A', there is one horizontal 1-cell from A to A' for each object M of $\mathbf{M}$

2-Cells, Identities, and Compositions A 2-cell from $A_0 \xrightarrow{M_1} A_1 \ldots A_{n-1} \xrightarrow{M_n} A_n$ to $A_0 \xrightarrow{M'} A_n$ (necessarily along vid_{A_0} and vid_{A_n}) is a multimorphism $m : [M_1, \ldots, M_n] \to M'$ of $\mathbf{M}$, and 2-identities and 2-compositions are inherited from $\mathbf{M}$ in the obvious manner

Definition (Enriched Category over a Path-Multicategory)**.** Given a Path-multicategory $\mathbf{P}$, a $\mathbf{P}$-enriched category is a set $\mathcal{C}_0$ and a Path-multifunctor $\mathbf{C} : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{1}) \to \mathbf{P}$.

Remark. Just as an enrichment over a monoidal category is a special case of an enrichment over a multicategory, an enrichment over a multicategory $\mathbf{M}$ is simply an enrichment over the Path-multicategory $\mathbf{Susp}_{\mathbb{1}}(\mathbf{M})$. In general, Path-multicategories are considered the most natural setting for enriched categories. But this construction also suggests another path for generalization, which we define next.

Definition (Enriched Classified Category over a Path-Multicategory)**.** Given a multicategory $\mathbf{E}$ and a Path-multicategory $\mathbf{P}$, a $\mathbf{P}$-enriched $\mathbf{E}$-classified category is a set $\mathcal{C}_0$ and a Path-multifunctor $\mathbf{C} : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{P}$.

Definition (Classified Category)**.** Given a multicategory $\mathbf{E}$, an $\mathbf{E}$-classified category is a set Ob "of objects" and a Path-multifunctor $\mathbf{C} : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{Susp}_{\mathbb{1}}(\mathbf{Set})$.

Example. Suppose $\mathbf{E}$ is a multipreorder describing an effect system, so that its objects are effects ε and its morphisms exist when a sequence of effects compose into a particular effect. Then a $\mathbf{E}$-classified category is essentially a $\mathbf{E}$-effectful language. Its objects can be viewed as types τ. For every pair of types τ and τ' and every effect ε, it assigns a set of morphisms, i.e. programs with effect ε, from τ to τ'. And for every sequence of effectful programs $f_1 : \tau_0 \xrightarrow{\varepsilon_1} \tau_1, \ldots, f_n : \tau_{n-1} \xrightarrow{\varepsilon_n} \tau_n$ and effect ε' such that $[\varepsilon_1, \ldots, \varepsilon_n] \leq \varepsilon'$, it assigns a composed ε'-effectful program $f_1 ; \ldots ; f_n : \tau_0 \xrightarrow{\varepsilon'} \tau_n$. In particular, for every type τ and effect ε such that $[\,] \leq \varepsilon$, it assigns an identity program with effect ε from τ to τ. Lastly, composition is well-behaved in the sense we are used to from normal categories.

Remark. Given a function $F_0 : \mathcal{C}_0 \to \mathcal{C}'_0$ and a multifunctor $F_E : \mathbf{E} \to \mathbf{E}'$ there is an obvious Path-multifunctor $F_S(F_0, F_E) : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{Susp}_{\mathcal{C}'_0}(\mathbf{E}')$. Thus we can define an enriched classified functor from a $\mathbf{P}$-enriched $\mathbf{E}$-classified category $\langle \mathcal{C}_0, \mathbf{C} : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{P}\rangle$ to a $\mathbf{P}$'-enriched $\mathbf{E}$'-classified category $\langle \mathcal{C}'_0, \mathbf{C}' : \mathbf{Susp}_{\mathcal{C}'_0}(\mathbf{E}') \to \mathbf{P}'\rangle$ as a function $F_0 : \mathcal{C}_0 \to \mathcal{C}'_0$ along with a multifunctor $F_E : \mathbf{E} \to \mathbf{E}'$, a Path-multifunctor $F_P : \mathbf{P} \to \mathbf{P}'$, and a natural transformation of Path-multifunctors $F_\theta : \mathbf{C}\,; F_P \Rightarrow F_S(F_0, F_E)\,; \mathbf{C}' : \mathbf{Span}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{P}'$. Note that when F_E is the identity multifunctor on $\mathbf{E}$, then we more specifically call an enriched classified functor $\mathbf{E}$-*classified*, and/or when F_P is the identity Path-multifunctor on $\mathbf{P}$, then enriched classified functors are more specifically called $\mathbf{P}$-*enriched*.

Remark. Because enriched classified functors can change enrichment and classification schemes, the structure of enriched classified natural transformations is complex. However, when the enrichment and classification scheme are fixed, i.e. we have a $\mathbf{P}$-enriched $\mathbf{E}$-classified natural transformation, then the structure can be concisely described, as we show on the next page. But even then, composition of natural transformations is non-trivial. Surprisingly, the seemingly more complex horizontal composition of natural transformations arises naturally, whereas vertical composition does not always exist. Fortunately, if the classification scheme $\mathbf{E}$ has a unit object, i.e. a "pure" effect, then one can generalize the familiar definition of vertical composition to $\mathbf{P}$-enriched $\mathbf{E}$-classified natural transformations.

Example. Just as a Lawvere metric space is an $\mathbf{R}^{\geq\infty}_{+\geq}$-enriched category, a metric map is an $\mathbf{R}^{\geq\infty}_{+\geq}$-enriched functor, and a 2-cell in **LMet** is an $\mathbf{R}^{\geq\infty}_{+\geq}$-enriched natural transformation (defined on the next page).

Definition (**P**-Enriched **E**-Classified Category **C**)**.** This is simply a more explicit version of the earlier definition:

Objects A set of $\mathcal{C}_0$ of objects, and for every C in $\mathcal{C}_0$ a 0-cell $\mathbf{C}(C)$ of $\mathbf{P}$

Hom-Sets For every pair of objects C and C' in $\mathcal{C}_0$ and object ε of $\mathbf{E}$, a horizontal 1-cell $\mathbf{C}_\varepsilon(C, C') : \mathbf{C}(C) \to \mathbf{C}(C')$ of $\mathbf{P}$

Compositions For every alternating series $C_0 \xrightarrow{\varepsilon_1} C_1 \ldots C_{n-1} \xrightarrow{\varepsilon_n} C_n$ of objects in $\mathcal{C}_0$ and objects of $\mathbf{E}$, and for every multimorphism $e : [\varepsilon_1, \ldots, \varepsilon_n] \to \varepsilon'$ of $\mathbf{E}$, a 2-cell of $\mathbf{P}$ as follows:

$$\begin{tikzcd}
\mathbf{C}(C_0) \arrow[rrrrr, "\mathbf{C}_{\varepsilon'}(C_0, C_n)"] & & & & & \mathbf{C}(C_n) \\
\mathbf{C}(C_0) \arrow[u, "id_{\mathbf{C}(C_0)}"] \arrow[r, "\mathbf{C}_{\varepsilon_1}(C_0, C_1)"'] & \mathbf{C}(C_1) \arrow[r] & \cdots \arrow[u, phantom, "\Uparrow \mathbf{C}_{C_0,\ldots,C_n}(e)"] \arrow[r] & \mathbf{C}(C_{n-1}) \arrow[rr, "\mathbf{C}_{\varepsilon_n}(C_{n-1}, C_n)"'] & & \mathbf{C}(C_n) \arrow[u, "id_{\mathbf{C}(C_n)}"']
\end{tikzcd}$$

Identity For every pair of objects C and C' in $\mathcal{C}_0$ and object ε of $\mathbf{E}$,

$$\begin{tikzcd}
\mathbf{C}(C) \arrow[r, "\mathbf{C}_\varepsilon(C, C')"] & \mathbf{C}(C') \\
\mathbf{C}(C) \arrow[u, "id_{\mathbf{C}(C)}"] \arrow[r, "\mathbf{C}_\varepsilon(C, C')"'] \arrow[u, phantom, shift left=8ex, "\Uparrow id_{\mathbf{C}_\varepsilon(C,C')}"] & \mathbf{C}(C') \arrow[u, "id_{\mathbf{C}(C')}"']
\end{tikzcd}
\quad \text{equals} \quad
\begin{tikzcd}
\mathbf{C}(C) \arrow[r, "\mathbf{C}_\varepsilon(C, C')"] & \mathbf{C}(C') \\
\mathbf{C}(C) \arrow[u, "id_{\mathbf{C}(C)}"] \arrow[r, "\mathbf{C}_\varepsilon(C, C')"'] \arrow[u, phantom, shift left=8ex, "\Uparrow \mathbf{C}_{C,C'}(id_\varepsilon)"] & \mathbf{C}(C') \arrow[u, "id_{\mathbf{C}(C')}"']
\end{tikzcd}$$

Associativity For every applicable tree (with most labels omitted for space),

$$\begin{tikzcd}[column sep=small]
{} \arrow[rrrrrrr] \arrow[drrrrrrr, phantom, "\Uparrow \mathbf{C}_{C_0^1, C_1^{m_1}, \ldots, C_n^{m_n}}(e')"] & & & & & & & {} \\
{} \arrow[u] \arrow[rrr] \arrow[drrr, phantom, "\Uparrow \mathbf{C}_{C_0^1, C_1^1, \ldots, C_1^{m_1}}(e_1)"] & & & {} \arrow[r, "\cdots" description] & {} \arrow[rrr] & & & {} \arrow[u] \\
{} \arrow[u] \arrow[r] & {} \arrow[r, "\cdots" description] & {} \arrow[r] & {} \arrow[u] \arrow[r, "\cdots" description] \arrow[ur, phantom, "\vdots"] & {} \arrow[u] \arrow[r] \arrow[urrr, phantom, "\Uparrow \mathbf{C}_{C_{n-1}^{m_{n-1}}, C_n^1, \ldots, C_n^{m_n}}(e_n)"] & {} \arrow[r, "\cdots" description] & {} \arrow[r] & {} \arrow[u]
\end{tikzcd}$$

equals

$$\begin{tikzcd}[column sep=small]
{} \arrow[rrrrrrr] \arrow[drrrrrrr, phantom, "\Uparrow \mathbf{C}_{C_0^1, C_1^1, \ldots, C_1^{m_1}, \ldots, C_{n-1}^{m_{n-1}}, C_n^1, \ldots, C_n^{m_n}}\left(\Delta_{e_1,\ldots,e_n} e'\right)"] & & & & & & & {} \\
{} \arrow[u] \arrow[r] & {} \arrow[r, "\cdots" description] & {} \arrow[r] & {} \arrow[r, "\cdots" description] & {} \arrow[r] & {} \arrow[r, "\cdots" description] & {} \arrow[r] & {} \arrow[u]
\end{tikzcd}$$

Definition (**P**-Enriched **E**-Classified Functor)**.** This is simply a more explicit version of the earlier definition of **P**-enriched **E**-classified functor F from a **P**-enriched **E**-classified category $\mathbf{C}$ with object set $\mathcal{C}_0$ to a **P**-enriched **E**-classified category $\mathbf{C}'$ with object set $\mathcal{C}'_0$:

Objects For each object C in $\mathcal{C}_0$, an object $F(C)$ in $\mathcal{C}'_0$

Vertical 1-Cells For each object C in $\mathcal{C}_0$, a vertical 1-cell $f_C : \mathbf{C}(C) \to \mathbf{C}'(F(C))$ of $\mathbf{P}$

2-Cells For each pair of objects C and C' in $\mathcal{C}_0$ and object ε of $\mathbf{E}$, a 2-cell of $\mathbf{P}$

$$\begin{tikzcd}
\mathbf{C}'(F(C)) \arrow[r, "\mathbf{C}'_{\varepsilon}(F(C),F(C'))"] & \mathbf{C}'(F(C')) \\
\mathbf{C}(C) \arrow[u, "f_C"] \arrow[r, "\mathbf{C}_{\varepsilon}(C,C')"'] \arrow[ur, phantom, "\Uparrow F_{\varepsilon}(C,C')"] & \mathbf{C}(C') \arrow[u, "f_{C'}"']
\end{tikzcd}$$

Preservation of Compositions For every alternating series $C_0 \xrightarrow{\varepsilon_1} C_1 \dots C_{n-1} \xrightarrow{\varepsilon_n} C_n$ of objects in $\mathcal{C}_0$ and objects of $\mathbf{E}$, and for every multimorphism $e : [\varepsilon_1, \dots, \varepsilon_n] \to \varepsilon'$,

$$\begin{tikzcd}
\mathbf{C}'(F(C_0)) \arrow[rrrr, "\mathbf{C}'_{\varepsilon'}(F(C_0),F(C_n))"] & & & & \mathbf{C}'(F(C_n)) \\
\mathbf{C}'(F(C_0)) \arrow[u, "id"] \arrow[r, "\mathbf{C}'_{\varepsilon_1}(F(C_0),F(C_1))"] & \mathbf{C}'(F(C_1)) \arrow[r, dash] & \cdots \arrow[u, phantom, "\Uparrow \mathbf{C}'_{F(C_0),\dots,F(C_n)}(e)"] \arrow[r] & \mathbf{C}'(F(C_{n-1})) \arrow[r, "\mathbf{C}'_{\varepsilon_n}(F(C_{n-1}),F(C_n))"] & \mathbf{C}'(F(C_n)) \arrow[u, "id"'] \\
\mathbf{C}(C_0) \arrow[u, "f_{C_0}"] \arrow[r, "\mathbf{C}_{\varepsilon_1}(C_0,C_1)"'] \arrow[ur, phantom, "\Uparrow F_{\varepsilon_1}(C_0,C_1)"] & \mathbf{C}(C_1) \arrow[u, "f_{C_1}"'] \arrow[r, dash] & \cdots \arrow[u, phantom, "\vdots"] \arrow[r] & \mathbf{C}(C_{n-1}) \arrow[u, "f_{C_{n-1}}"] \arrow[r, "\mathbf{C}_{\varepsilon_n}(C_{n-1},C_n)"'] \arrow[ur, phantom, "\Uparrow F_{\varepsilon_n}(C_{n-1},C_n)"] & \mathbf{C}(C_n) \arrow[u, "f_{C_n}"']
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathbf{C}'(F(C_0)) \arrow[rrrr, "\mathbf{C}'_{\varepsilon'}(F(C_0),F(C_n))"] & & & & \mathbf{C}'(F(C_n)) \\
\mathbf{C}(C_0) \arrow[u, "f_{C_0}"] \arrow[rrrr, "\mathbf{C}_{\varepsilon'}(C_0,C_n)" description] & & \arrow[u, phantom, "\Uparrow F_{\varepsilon'}(C_0,C_n)"] & & \mathbf{C}(C_n) \arrow[u, "f_{C_n}"'] \\
\mathbf{C}(C_0) \arrow[u, "id"] \arrow[r, "\mathbf{C}_{\varepsilon_1}(C_0,C_1)"'] & \mathbf{C}(C_1) \arrow[r, dash] & \cdots \arrow[u, phantom, "\Uparrow \mathbf{C}_{C_0,\dots,C_n}(e)"] \arrow[r] & \mathbf{C}(C_{n-1}) \arrow[r, "\mathbf{C}_{\varepsilon_n}(C_{n-1},C_n)"'] & \mathbf{C}(C_n) \arrow[u, "id"']
\end{tikzcd}$$

Definition (**P**-Enriched **E**-Classified Natural Transformation)**.** A **P**-enriched **E**-classified natural transformation α from a **P**-enriched **E**-classified functor F to a **P**-enriched **E**-classified functor G both from a **P**-enriched **E**-classified category $\mathbf{C}$ with object set $\mathcal{C}_0$ to a **P**-enriched **E**-classified category $\mathbf{C}'$ with object set $\mathcal{C}'_0$ is comprised of the following:

2-Cells For each pair of objects C and C' in $\mathcal{C}_0$ and object ε of $\mathbf{E}$, a 2-cell of $\mathbf{P}$

$$\begin{tikzcd}
F(C) \arrow[r, "\mathbf{C}'_{\varepsilon}(F(C),G(C'))"] & G(C') \\
C \arrow[u, "f_C"] \arrow[r, "\mathbf{C}_{\varepsilon}(C,C')"'] \arrow[ur, phantom, "\Uparrow \alpha_{\varepsilon}(C,C')"] & C' \arrow[u, "g_{C'}"']
\end{tikzcd}$$

Naturality For every alternating series $C_0 \xrightarrow{\varepsilon_1} C_1 \dots C_{n-1} \xrightarrow{\varepsilon_n} C_n$ of objects in $\mathcal{C}_0$ and objects of $\mathbf{E}$, and for every multimorphism $e : [\varepsilon_1, \dots, \varepsilon_n] \to \varepsilon'$ and index i in $\{1, \dots, n\}$,

$$\begin{tikzcd}
\mathbf{C}'(F(C_0)) \arrow[rrrrrr, "\mathbf{C}'_{\varepsilon'}(F(C_0),G(C_n))"] & & & & & & \mathbf{C}'(G(C_n)) \\
\mathbf{C}'(F(C_0)) \arrow[u, "id"] \arrow[r] & \mathbf{C}'(F(C_1)) \arrow[r, phantom, "\cdots"] & \mathbf{C}'(F(C_{i-1})) \arrow[r] & \mathbf{C}'(G(C_i)) \arrow[u, phantom, "\Uparrow \mathbf{C}'_{F(C_0),\dots,F(C_{i-1}),G(C_i),G(C_n)}(e)"] \arrow[r, phantom, "\cdots"] & \mathbf{C}'(G(C_{n-1})) \arrow[r] & & \mathbf{C}'(G(C_n)) \arrow[u, "id"'] \\
\mathbf{C}(C_0) \arrow[u, "f_{C_0}"] \arrow[r, "\mathbf{C}_{\varepsilon_1}(C_0,C_1)"'] \arrow[ur, phantom, "\Uparrow F_{\varepsilon_1}(C_0,C_1)"] & \mathbf{C}(C_1) \arrow[u, "f_{C_1}"'] \arrow[r, dashed] & \mathbf{C}(C_{i-1}) \arrow[u, "f_{C_{i-1}}"] \arrow[r, "\mathbf{C}_{\varepsilon}(C_{i-1},C_i)"'] \arrow[ur, phantom, "\Uparrow \alpha_{\varepsilon_i}(C_{i-1},C_i)"] & \mathbf{C}(C_i) \arrow[u, "g_{C_i}"'] \arrow[r, dashed] & \mathbf{C}(C_{n-1}) \arrow[u, "g_{C_{n-1}}"] \arrow[rr, "\mathbf{C}_{\varepsilon_n}(C_{n-1},C_n)"'] & \arrow[u, phantom, "\Uparrow F_{\varepsilon_n}(C_{n-1},C_n)"] & \mathbf{C}(C_n) \arrow[u, "g_{C_n}"']
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\mathbf{C}'(F(C_0)) \arrow[rrrrrr, "\mathbf{C}'_{\varepsilon'}(F(C_0),G(C_n))"] & & & & & & \mathbf{C}'(G(C_n)) \\
\mathbf{C}(C_0) \arrow[u, "f_{C_0}"] \arrow[rrrrrr] & & & \arrow[u, phantom, "\Uparrow \alpha_{\varepsilon'}(C_0,C_n)"] & & & \mathbf{C}(C_n) \arrow[u, "g_{C_n}"'] \\
\mathbf{C}(C_0) \arrow[u, "id"] \arrow[r, "\mathbf{C}_{\varepsilon_1}(C_0,C_1)"'] & \mathbf{C}(C_1) \arrow[r, dashed] & \mathbf{C}(C_{i-1}) \arrow[r, "\mathbf{C}_{\varepsilon_i}(C_{i-1},C_i)"'] \arrow[ur, phantom, "\Uparrow \mathbf{C}_{C_0,\dots,C_n}(e)"] & \mathbf{C}(C_i) \arrow[r, dashed] & \mathbf{C}(C_{n-1}) \arrow[rr, "\mathbf{C}_{\varepsilon_n}(C_{n-1},C_n)"'] & & \mathbf{C}(C_n) \arrow[u, "id"']
\end{tikzcd}$$

Definition (Transformations with Changes of Enrichment and Classification)**.** Given a $\mathbf{P}$-enriched $\mathbf{E}$-classified category $\langle \mathcal{C}_0, \mathbf{C} : \mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E}) \to \mathbf{P}\rangle$ and a $\mathbf{P}$'-enriched $\mathbf{E}$'-classified category $\langle \mathcal{C}_0', \mathbf{C}' : \mathbf{Susp}_{\mathcal{C}_0'}(\mathbf{E}') \to \mathbf{P}'\rangle$, define the *lax relational* **Path**-multicategory $\mathbf{Trans}(\langle \mathcal{C}_0, \mathbf{C}\rangle, \langle \mathcal{C}_0', \mathbf{C}'\rangle)$ as follows:

Objects An object M assigns to each object P of $\mathbf{P}$ an object $M(P')$ of $\mathbf{P}'$, and to each object C in $\mathcal{C}_0$ an object $M(C)$ in $\mathcal{C}_0$ and a vertical 1-cell $m_C : M(\mathbf{C}(C)) \to \mathbf{C}'(M(C))$ of $\mathbf{P}'$.

Vertical 1-Cells A vertical 1-cell F from M to M' assigns to each vertical 1-cell $v : P \to P'$ of $\mathbf{P}$ a vertical 1-cell $F(v) : M(P) \to M'(P')$ of $\mathbf{P}'$, such that for every object C in $\mathcal{C}_0$, the object $M(C)$ equals $M'(C)$ in $\mathcal{C}_0'$ and the vertical morphism m_C equals $F(id_{\mathbf{C}(C)})\,;m'_C$ in $\mathbf{P}'$.

Horizontal 1-Cells A horizontal 1-cell T from M to M' assigns to each horizontal 1-cell $h : P \to P'$ of $\mathbf{P}$ a horizontal 1-cell $T(h) : M(P) \to M'(P')$ of $\mathbf{P}'$, and to each object ε of $\mathbf{E}$ an object $T(\varepsilon)$ of $\mathbf{E}'$, and to each pair of objects C and C' in $\mathcal{C}_0$ and object ε of $\mathbf{E}$ a 2-cell of $\mathbf{P}'$

$$\begin{tikzcd}
\mathbf{C}'(M(C)) \arrow[r, "\mathbf{C}'_{T(\varepsilon)}(M(C), M'(C'))"] & \mathbf{C}'(M'(C')) \\
M(\mathbf{C}(C)) \arrow[u, "m_C"] \arrow[r, "T(\mathbf{C}_\varepsilon(C, C'))"'] \arrow[ur, phantom, "\Uparrow t_\varepsilon(C, C')"] & M'(\mathbf{C}(C')) \arrow[u, "m'_{C'}"']
\end{tikzcd}.$$

2-Cells A 2-cell Θ from $M^0 \xrightarrow{T^1} M^1 \cdots M^{n-1} \xrightarrow{T^n} M^n$ to $M' \xrightarrow{T'} M''$ along F' and F'' assigns to each 2-cell $\alpha \in \mathsf{Face}_{v',v''}([h_1, \ldots, h_n], h')$ of $\mathbf{P}$ a 2-cell $\Theta(\alpha) \in \mathsf{Face}_{F'(v'),F''(v'')}([T^1(h_1), \ldots, T^n(h_n)], T'(h'))$ of $\mathbf{P}'$, and to each multimorphism $e : [\varepsilon_1, \ldots, \varepsilon_n] \to \varepsilon'$ of $\mathbf{E}$ a multimorphism $\Theta(e) : [T^1(\varepsilon_1), \ldots, T^n(\varepsilon_n)] \to T'(\varepsilon')$ of $\mathbf{E}'$, such that for every list $C_0, \ldots, C_n$ of objects in $\mathcal{C}_0$ and every multimorphism $e : [\varepsilon_1, \ldots, \varepsilon_n] \to \varepsilon'$ of $\mathbf{E}$ the following holds in $\mathbf{P}'$:

$$\begin{tikzcd}[column sep=small]
\mathbf{C}'(M'(C_0)) \arrow[rrrr, "\mathbf{C}'_{T'(\varepsilon')}(M'(C_0), M''(C_n))"] \arrow[d, dash] & & & & \mathbf{C}'(M''(C_n)) \arrow[d, dash] \\
\mathbf{C}'(M^0(C_0)) \arrow[r] & \mathbf{C}'(M^1(C_1)) \arrow[r, dash] & \cdots \arrow[r] \arrow[u, phantom, "\Uparrow \mathbf{C}'_{M^0(C_0),\ldots,M^n(C_n)}(\Theta(e))"] & \mathbf{C}'(M^{n-1}(C_{n-1})) \arrow[r] & \mathbf{C}'(M^n(C_n)) \\
M^0(\mathbf{C}(C_0)) \arrow[u, "m^0_{C_0}"] \arrow[r, "T^1(\mathbf{C}_{\varepsilon_1}(C_0, C_1))"'] \arrow[ur, phantom, "\Uparrow t^1_{\varepsilon_1}(C_0, C_1)"] & M^1(\mathbf{C}(C_1)) \arrow[u, "m^1_{C_1}"'] \arrow[r, dash] & \cdots \arrow[r] \arrow[u, phantom, "\vdots"] & M^{n-1}(\mathbf{C}(C_{n-1})) \arrow[u, "m^{n-1}_{C_{n-1}}"] \arrow[r, "T^n(\mathbf{C}_{\varepsilon_n}(C_{n-1}, C_n))"'] \arrow[ur, phantom, "\Uparrow t^n_{\varepsilon_n}(C_{n-1}, C_n)"] & M^n(\mathbf{C}(C_n)) \arrow[u, "m^n_{C_n}"']
\end{tikzcd}$$

equals

$$\begin{tikzcd}[column sep=small]
\mathbf{C}'(M'(C_0)) \arrow[rrrr, "\mathbf{C}'_{T'(\varepsilon')}(M'(C_0), M''(C_n))"] & & & & \mathbf{C}'(M''(C_n)) \\
M'(\mathbf{C}(C_0)) \arrow[u, "m'_{C_0}"] \arrow[rrrr] & & {} \arrow[u, phantom, "\Uparrow t'_{\varepsilon'}(C_0, C_n)"] & & M''(\mathbf{C}(C_n)) \arrow[u, "m''_{C_n}"'] \\
M^0(\mathbf{C}(C_0)) \arrow[u, "F'(id_{\mathbf{C}(C_0)})"'] \arrow[r, "T^1(\mathbf{C}_{\varepsilon_1}(C_0, C_1))"'] & M^1(\mathbf{C}(C_1)) \arrow[r, dash] & \cdots \arrow[r] \arrow[u, phantom, "\Uparrow \Theta(\mathbf{C}_{C_0,\ldots,C_n}(e))"] & M^{n-1}(\mathbf{C}(C_{n-1})) \arrow[r, "T^n(\mathbf{C}_{\varepsilon_n}(C_{n-1}, C_n))"'] & M^n(\mathbf{C}(C_n)) \arrow[u, "F''(id_{\mathbf{C}(C_n)})"]
\end{tikzcd}$$

Vertical Composition A chain of vertical 1-cells $M_0 \xrightarrow{F_1} M_1 \cdots M_{n-1} \xrightarrow{F_n} M_n$ composes to a vertical 1-cell $M_0 \xrightarrow{F'} M_n$ when every chain of vertical 1-cells $P_0 \xrightarrow{v_1} P_1 \cdots P_{n-1} \xrightarrow{v_n} P_n$ in $\mathbf{P}$ has the property that the vertical 1-cell $F_1(v_1)\,; \ldots ; F_n(v_n)$ equals $F'(v_1\,; \ldots ; v_n)$ in $\mathbf{P}'$.

2-Composition Since we have not developed the appropriate notation, we describe this informally. It is the obvious adaptation of the concept employed above for vertical composition. A pasting diagram of 2-cells $\vec{\Theta}$ composes to a 2-cell Θ' when every similarly shaped pasting diagram of 2-cells $\vec{\alpha}$ in $\mathbf{P}$ has the property that the composition of the corresponding mapping, i.e. $\triangle \vec{\Theta}(\vec{\alpha})$, equals the corresponding mapping of the composition, i.e. $\Theta'(\triangle \vec{\alpha})$, in $\mathbf{P}'$ *and* every similarly shaped multigraph of multimorphisms $\vec{e}$ in $\mathbf{E}$ has the property that the composition of the corresponding mapping, i.e. $\triangle \vec{\Theta}(\vec{e})$, equals the corresponding mapping of the composition, i.e. $\Theta'(\triangle \vec{e})$, in $\mathbf{E}'$.

Remark. A 2-cell is an identity if it maps identity 2-cells of $\mathbf{P}$ to identity 2-cells of $\mathbf{P}'$ and identity multimorphisms of $\mathbf{E}$ to identity multimorphisms of $\mathbf{E}'$.

Definition (Enriched Classified Functor)**.** Consider what an internal monoid of $\mathbf{Trans}(\langle\mathcal{C}_0, \mathbf{C}\rangle, \langle\mathcal{C}'_0, \mathbf{C}'\rangle)$ is comprised of: a 0-cell M, a vertical 1-cell F on that 0-cell, a horizontal 1-cell T on that 0-cell, and a 2-cell Θ_n from $[T, \ldots (n \text{ times})]$ to T along F and F for each n in $\mathbb{N}$, altogether satisfying certain compositional properties. These various components can be broken down into mappings from $\mathbf{P}$ to $\mathbf{P}'$, mappings from $\mathbf{E}$ to $\mathbf{E}'$, and a mapping from $\mathcal{C}_0$ to $\mathcal{C}'_0$ and corresponding transformations in $\mathbf{P}'$. In fact, the compositional properties exactly describe the requirements for the mappings from $\mathbf{P}$ to $\mathbf{P}'$ to collectively form a Path-multifunctor F_P, and for the mappings from $\mathbf{E}$ to $\mathbf{E}'$ to collectively form a multifunctor F_E, which the mapping F_0 from $\mathcal{C}_0$ to $\mathcal{C}'_0$ extends to a Path-multifunctor $F_S(F_0, F_E)$ from $\mathbf{Susp}_{\mathcal{C}_0}(\mathbf{E})$ to $\mathbf{Susp}_{\mathcal{C}'_0}(\mathbf{E}')$, and for the transformations to collectively form a natural transformation F_θ from the resulting Path-multifunctor $\mathbf{C}\,; F_P$ to the resulting Path-multifunctor $F_S(F_0, F_E)\,; \mathbf{C}'$. In other words, an internal monoid of $\mathbf{Trans}(\langle\mathcal{C}_0, \mathbf{C}\rangle, \langle\mathcal{C}'_0, \mathbf{C}'\rangle)$ is precisely an enriched classified functor from $\langle\mathcal{C}_0, \mathbf{C}\rangle$ to $\langle\mathcal{C}'_0, \mathbf{C}'\rangle$.

Definition (Enriched Classified Natural Transformation)**.** An enriched classified natural transformation from an enriched classified functor to another is an internal bimodule of $\mathbf{Trans}(\langle\mathcal{C}_0, \mathbf{C}\rangle, \langle\mathcal{C}'_0, \mathbf{C}'\rangle)$ between the internal monoids corresponding to the enriched classified functors.

Example. Let $\mathbf{P}$ and $\mathbf{P}'$ both be $\mathbf{Set}$, and consider the cases where F_P above is the identity. In other words, consider only classified categories. Let $\mathbf{E}$ be $\mathbf{1}$, so that an $\mathbf{E}$-classified category is simply a category. Let $\mathbf{E}'$ be a multipreorder with a unit, i.e. an effect system with a pure effect. Let $\mathbf{C}'$ be a $\mathbf{E}'$-classified category, i.e. an effectful sequential language. Let $\mathbf{C}$ be the category of pure-classified morphisms of $\mathbf{E}'$ (which forms a category because a unit of a multicategory is an internal monoid of that multicategory). Then the inclusion of $\mathbf{C}$ into $\mathbf{C}'$ forms a classified functor I that changes classification from $\mathbf{1}$ to $\mathbf{E}'$ by mapping the unique object in $\mathbf{1}$ to the unit of $\mathbf{E}'$. Suppose furthermore that an object ε of $\mathbf{E}'$ represents a producer effect of $\mathbf{C}'$. Then there is an endofunctor F_ε on $\mathbf{C}$ and a family of ε-classified morphisms $\{\mathsf{exec}_\varepsilon : F_\varepsilon\tau \xrightarrow{\varepsilon} \tau \in \mathbf{C}'\}_{\tau\in\mathbf{C}}$ that forms a classified transformation from $F_\varepsilon\,; I$ to I.