Metric Spaces and Topologies

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February 12, 2018

1 Metric Spaces

Definition. $\mathbb{R}^{\geq\infty}$ is the set of non-negative real numbers along with the "value" ∞, i.e. $\mathbb{R}^{\geq} \cup \{\infty\}$. Addition, multiplication, and inequality are defined on ∞ as one would expect.

Definition. A Lawvere metric space, also known as a psuedoquasimetric or hemimetric space, is a set X and a "distance" function $d : X \times X \to \mathbb{R}^{\geq\infty}$ satisfying the following properties:

Point Inequality $\forall x \in X.\, 0 \geq d(x,x)$

Triangle Inequality $\forall x, y, z \in X.\, d(x,y) + d(y,z) \geq d(x,z)$

Definition. A metric space is a Lawvere metric space $\langle X, d\rangle$ satisfying the following additional properties:

Finiteness $\forall x, y \in X.\, d(x,y) \neq \infty$

Symmetry $\forall x, y \in X.\, d(x,y) = d(y,x)$

Separation $\forall x, y \in X.\, d(x,y) = 0 \implies x = y$

Example. Given a "dimension" D (which in general is just a set) and a "power" $p \geq 1$, define $d_p : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^{\geq\infty}$:

$$d_p(\vec{x}, \vec{y}) = \sqrt[p]{\sum_{i \in D} |y_i - x_i|^p}$$

The pair $\langle \mathbb{R}^D, d_p\rangle$ is a symmetric and separable Lawvere metric space that is also finite whenever D is finite. That is, $\langle \mathbb{R}^D, d_p\rangle$ is a metric space whenever D is finite. When p is 2, this is the Euclidean metric space, which is the "standard" metric on $\mathbb{R}^D$. When p is 1, this is the Manhattan metric space.

Example. Given a "dimension" D (which in general is just some set), define $d_{\max} : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^{\geq\infty}$ as follows:

$$d_{\max}(\vec{x}, \vec{y}) = \max_{i \in D} |y_i - x_i|$$

$\langle \mathbb{R}^D, d_{\max}\rangle$ is a symmetric, separable Lawvere metric space that is finite whenever D is finite. Note $d_{\max} = \lim_{p \to \infty} d_p$.

Example. Given a set of "outputs" O and a set of "inputs" I, define $d_0 : O^I \times O^I \to \mathbb{R}^{\geq\infty}$ as follows:

$$d_0(f, f') = |\{f(i) \neq f'(i) \mid i \in D\}| \quad \text{(i.e. the number of inputs on which } f \text{ and } f' \text{ differ)}$$

The pair $\langle O^I, d_0\rangle$ is a symmetric and separable Lawvere metric space that is also finite whenever I is finite.

Definition. A metric map (also known as a nonexpansive function or a weak contraction) from a (Lawvere) metric space $\langle X, d_X\rangle$ to a (Lawvere) metric space $\langle Y, d_Y\rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall x, x' \in X.\, d_X(x, x') \geq d_Y(f(x), f(x'))$$

Definition. A uniformly continuous function from a (Lawvere) metric space $\langle X, d_X\rangle$ to a (Lawvere) metric space $\langle Y, d_Y\rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall \varepsilon \in \mathbb{R}^{>}.\, \exists \delta \in \mathbb{R}^{>}.\, \forall x, x' \in X.\, d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

Definition. A continuous function from a (Lawvere) metric space $\langle X, d_X\rangle$ to a (Lawvere) metric space $\langle Y, d_Y\rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall x \in X.\, \forall \varepsilon \in \mathbb{R}^{>}.\, \exists \delta \in \mathbb{R}^{>}.\, \forall x' \in X.\, d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

Example. $\lambda x.\, 2x : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function on $\langle \mathbb{R}, d_2\rangle$ that is not a metric map. $\lambda x.\, x^{-1} : \mathbb{R}^{>} \to \mathbb{R}^{>}$ is a continuous function on $\langle \mathbb{R}, d_2\rangle$ that is not uniformly continuous.

Definition. $(\mathbf{L})\mathbf{Met}$ is the category of (Lawvere) metric spaces and metric maps. $(\mathbf{L})\mathbf{Met}_u$ is the category of (Lawvere) metric spaces and uniformly continuous functions. $(\mathbf{L})\mathbf{Met}_c$ is the category of (Lawvere) metric spaces and continuous functions.

2 Topologies

Definition. A topological space is a set X along with a "topology" $\tau \subseteq \mathbb{P}X$ of "open" sets that is closed under finite intersections and arbitrary unions, meaning it satisfies the following properties:

- $X \in \tau$
- $\forall O, O' \in \tau.\ O \cap O' \in \tau$
- $\forall \mathcal{O} \subseteq \tau.\ \bigcup \mathcal{O} \in \tau$

Example. The Sierpiński space S is the pair $\langle \mathbb{B}, \{\varnothing, \{\mathbb{t}\}, \mathbb{B}\}\rangle$.

Definition. Given a function $f : X \to Y$ and a subset $\mathcal{Y}$ of Y, the preimage of $\mathcal{Y}$ under f, which is denoted $f^{-1}(\mathcal{Y})$, is the set $\{x \in X \mid f(x) \in \mathcal{Y}\}$.

Definition. A continuous function from a topological space $\langle X, \tau_X\rangle$ to a topological space $\langle Y, \tau_Y\rangle$ is a function $f : X \to Y$ whose preimage function preserves openness, meaning the following property holds:

$$\forall O \in \tau_Y.\ f^{-1}(O) \in \tau_X$$

Definition. Top is the category of topologic spaces and continuous functions.

3 Functors

Definition. Every metric map is a uniformly continuous function, and every uniformly continuous function is a continuous function, so there are "obvious" inclusion functors $(\mathbf{L})\mathbf{Met} \hookrightarrow (\mathbf{L})\mathbf{Met}_u \hookrightarrow (\mathbf{L})\mathbf{Met}_c$.

Definition. $\mathbb{R}^{>\infty}$ is the set of *positive* real numbers along with the "value" ∞, i.e. $\mathbb{R}^{\geq} \cup \{\infty\}$.

Definition. For a (Lawvere) metric space $\langle X, d\rangle$, the open ball of radius $r \in \mathbb{R}^{>\infty}$ around an element x of X, denoted $B_r(x)$, is the set $\{x' \in X \mid d(x, x') < r\}$.

Definition. Given a (Lawvere) metric space $\langle X, d\rangle$, the corresponding topology τ_d is the set of all subsets $O \subseteq X$ satisfying the following property:

$$\forall x \in O.\ \exists r \in \mathbb{R}^{>\infty}.\ B_r(x) \subseteq O$$

Definition. As an abuse of notation, $\tau : (\mathbf{L})\mathbf{Met}_{(u/c)} \to \mathbf{Top}$ is the functor mapping a (Lawvere) metric space $\langle X, d\rangle$ to the topological space $\langle X, \tau_d\rangle$, and mapping a nonexpansive/(uniformly)-continuous function f to itself, since it can be proven to be a continuous function between the corresponding topological spaces.

Example. Define $d_S : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ such that $d(\mathbb{t}, \mathbb{f})$ maps to 1 and all other inputs map to 0. Then the evaluation of $\tau(\langle \mathbb{B}, d_S\rangle)$ is the Sierpi nski space S.