Monoidal Categories

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Definition. Given a category $\mathbf{C}$, the category $\mathbb{L}\mathbf{C}$ is comprised of the following:

Objects An object is a list of objects of C.

Morphisms A morphism from $[A_1, \ldots, A_m]$ to $[B_1, \ldots, B_n]$ only exists when m equals n, in which case it is a list of morphisms $f_1 : A_1 \to B_1, \ldots, f_n : A_n \to B_n$ of $\mathbf{C}$

Identities The identity of $[A_1, \ldots, A_n]$ is $[id_{A_1}, \ldots, id_{A_n}]$

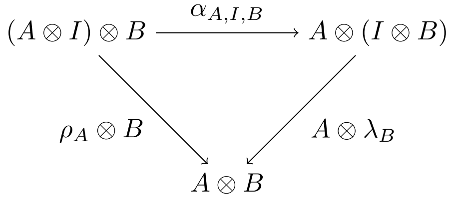
Composition The composition of $[f_1, \ldots, f_n]$ with $[g_1, \ldots, g_n]$ is $[f_1 \,;g_1, \ldots, f_n \,; g_n]$

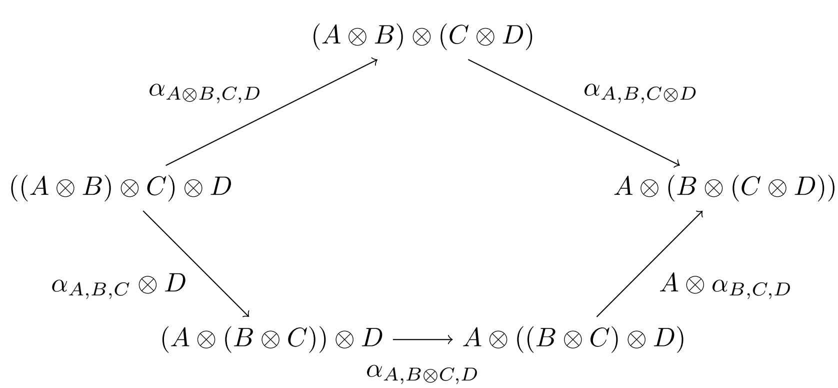
Remark. The above construction extends to a 2-monad *on* $\mathbf{Cat}$, with $\eta_{\mathbf{C}}$ being the *functor* that maps each object and each morphism to the singleton list containing it, and $\mu_{\mathbf{C}}$ being the functor that maps each list of lists of objects and each list of lists of morphisms to its flattening.

Definition (Strict Monoidal Category)**.** A strict monoidal category is equivalently defined as a (strict) monad algebra of $\mathbb{L}$ on $\mathbf{Cat}$ or as an internal monoid of the multicategory $\mathbf{Cat}$, i.e. a category $\mathbf{C}$ along with an object I of $\mathbf{C}$ and a binary functor $\otimes : [\mathbf{C}, \mathbf{C}] \to \mathbf{C}$ satisfying identity and associativity. Similarly, a strict monoidal functor is equivalently defined as a (strict) morphism of monad algebras of $\mathbb{L}$ on $\mathbf{Cat}$ or as an internal monoid homomorphism of the multicategory $\mathbf{Cat}$, i.e. a functor F that preserves I and $\otimes$ (strictly). A strict monoidal transformation is defined as a transformation of (strict) morphisms of monad algebras of $\mathbb{L}$ on $\mathbf{Cat}$, or equivalently as a natural transformation α from F to G such that α_I equals $id_{I'}$ and $\alpha_{A \otimes B}$ equals $\alpha_A \otimes' \alpha_B$.

Definition (Weak Monoidal Category)**.** A weak monoidal category is equivalently defined as a weak monad algebra of $\mathbb{L}$ on $\mathbf{Cat}$ or as the category of unary morphisms of a representable multicategory. Similarly, a weak monoidal functor is equivalently defined as a weak morphism of weak monad algebras of $\mathbb{L}$ on $\mathbf{Cat}$ or as a functor corresponding to a tensor-preserving multifunctor between representable multicategories, i.e. multifunctors F with the property that FT with Ft is a tensor of $FA_1, \ldots, FA_n$ whenever T with t is a tensor of $A_1, \ldots, A_n$. A weak monoidal transformation is defined as a transformation of weak morphisms of weak monad algebras of $\mathbb{L}$ on $\mathbf{Cat}$, or equivalently as a natural fransformation corresponding to a natural transformation α of tensor-preserving multifunctors such that $\alpha_T : FT \to GT$, where T with $t : \vec{A} \to T$ is a tensor of $A_1, \ldots, A_n$, has the property that $\triangle_{Ft} \alpha_T$ equals $\triangle_{\alpha_{A_1}, \ldots, \alpha_{A_n}} Gt$.

Remark. Yet another equivalent, and particularly common, definition of weak monoidal category is a category $\mathbf{C}$ along with an object I of $\mathbf{C}$, a binary functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, and natural isomorphisms $\{\rho_A : A \otimes I \to A\}_{A \in \mathbf{C}}$, $\{\lambda_A : I \otimes A \to A\}_{A \in \mathbf{C}}$, and $\{\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)\}_{A,B,C \in \mathbf{C}}$ making the following triangle and pentagon commute:





Definition (Lax Monoidal Functor)**.** A lax monoidal functor between strict/weak monoidal categories is equivalently defined as a lax morphism of strict/weak monad algebras of $\mathbb{L}$ on **Cat** or as a functor corresponding to a (not necessarily tensor-preserving) multifunctor between representable multicategories. A lax monoidal transformation is defined as a transformation of lax morphisms of weak monad algebras of $\mathbb{L}$ on **Cat**, or equivalently as a natural fransformation corresponding to a natural transformation of multifunctors.

Remark. Yet another equivalent, and particularly common, definition of lax monoidal functor corresponding to the earlier common definition of weak monoidal categories is a functor F along with a morphism $\mathsf{merge} : I' \to FI$ and a natural transformation $\{\mathsf{merge}_{A,B} : FA \otimes' FB \to F(A \otimes B)\}_{A,B \in \mathbf{C}}$ such that the following diagrams commute:

$$\begin{array}{ccc} I' \otimes' FA & \xrightarrow{\mathsf{merge} \otimes' FA} & FI \otimes' FA \\ {\scriptstyle \lambda'_{FA}} \downarrow & & \downarrow {\scriptstyle \mathsf{merge}_{I,A}} \\ FA & \xleftarrow[F\lambda_A]{} & F(I \otimes A) \end{array} \qquad \begin{array}{ccc} FA \otimes' I' & \xrightarrow{FA \otimes' \mathsf{merge}} & FA \otimes' FI \\ {\scriptstyle \rho'_{FA}} \downarrow & & \downarrow {\scriptstyle \mathsf{merge}_{A,I}} \\ FA & \xleftarrow[F\rho_A]{} & F(A \otimes I) \end{array}$$

$$\begin{array}{ccc} (FA \otimes' FB) \otimes' FC & \xrightarrow{\alpha'_{FA,FB,FC}} & FA \otimes' (FB \otimes' FC) \\ {\scriptstyle \mathsf{merge}_{A,B} \otimes' FC} \downarrow & & \downarrow {\scriptstyle FA \otimes' \mathsf{merge}_{B,C}} \\ F(A \otimes B) \otimes' FC) & & FA \otimes' F(B \otimes C) \\ {\scriptstyle \mathsf{merge}_{A \otimes B,C}} \downarrow & & \downarrow {\scriptstyle \mathsf{merge}_{A,B \otimes C}} \\ F((A \otimes B) \otimes C) & \xrightarrow[F\alpha_{A,B,C}]{} & F(A \otimes (B \otimes C)) \end{array}$$