Weighted Limits and Colimits

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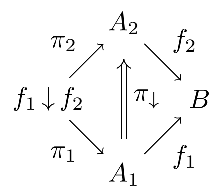
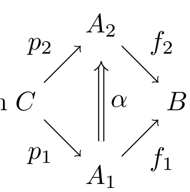
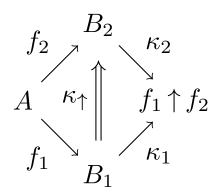
Definition (Comma Object)**.** Given 1-cells $A_1 \xrightarrow{f_1} B \xleftarrow{f_2} A_2$ of a 2-category, a comma object from f_1 to f_2 is a 0-cell, typically denoted $f_1 \downarrow f_2$, along with 1- and 2-cells as in the following diagram

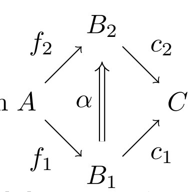


that is *universal* in the sense that given any other diagram  there exists a unique 1-cell $\langle\alpha\rangle : C \to f_1 \downarrow f_2$ such that $\langle\alpha\rangle * \pi_\downarrow$ equals α (and $\langle\alpha\rangle\,;\pi_1$ equals p_1 and $\langle\alpha\rangle\,;\pi_2$ equals p_2).

Example. Comma categories are the comma objects of **Cat**.

Definition (Cocomma Object)**.** Given 1-cells $B_1 \xleftarrow{f_1} A \xrightarrow{f_2} B_2$ of a 2-category, a cocomma object from f_1 to f_2 is a 0-cell, typically denoted $f_1 \uparrow f_2$, along with 1- and 2-cells as in the following diagram



that is *universal* in the sense that given any other diagram  there exists a unique 1-cell $[\alpha] : f_1 \uparrow f_2 \to C$ such that $\kappa_\uparrow * [\alpha]$ equals α (and $\kappa_1\,;[\alpha]$ equals c_1 and $\kappa_2\,;[\alpha]$ equals c_2).

Example. For the 1-source $1 \xleftarrow{!} A \xrightarrow{id} A$ in **Prost**, the corresponding cocomma object $!\uparrow A$ is the set Option(A) with none being smaller than some(a) for all $a \in A$. On the flipside, the cocomma object $A \uparrow !$ is the set Option(A) with none being larger than some(a) for all $a \in A$. In both cases, some(a) is less than some(a') iff a is less than a'.

Note that $\mathbb{L}(A)$ in **Set** can be defined as the fixpoint $\mu X.1 + (A \times X)$. In **Prost**, the fixpoints $\mu X.1 + (A \times X)$, $\mu X.!\uparrow(A \times X)$, and $\mu X.(A \times X)\uparrow!$ all correspond to lists but with different orderings. In the first, $\ell \leq \ell'$ can only hold if ℓ and ℓ' have the same length, whereas in the second ℓ can be a prefix of ℓ', and in the third ℓ' can be a prefix of ℓ. In particular, they all agree on lists with the same length, in which case they use componentwise comparison; where they differ is how they handle lists of differing length.

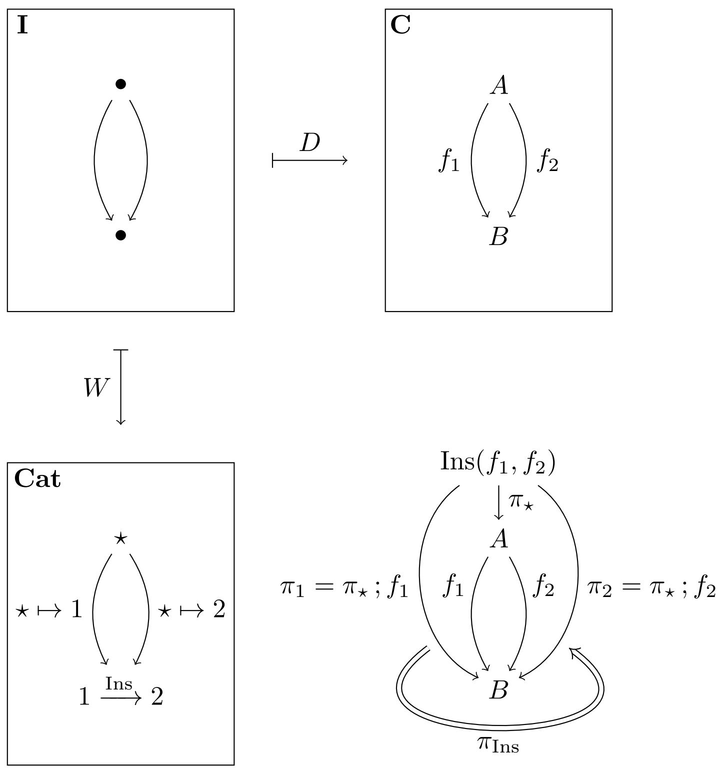
Definition (Inserter Object)**.** Given two 1-cells $f_1, f_2 : A \to B$ of a 2-category, an inserter from f_1 to f_2 is a 0-cell, typically denoted $\mathrm{Ins}(f_1, f_2)$, along with 1-cell $\pi : \mathrm{Ins}(f_1, f_2) \to A$ and 2-cell $\pi_{\mathrm{Ins}} : \pi\,; f_1 \Rightarrow \pi\,; f_2 : \mathrm{Ins}(f_1, f_2) \to B$ that is *universal*, meaning given any other 0-cell C with 1-cell $f : C \to A$ and 2-cell $\alpha : f\,; f_1 \Rightarrow f\,; f_2 : C \to B$ there exists a unique 1-cell $\langle\alpha\rangle : C \to \mathrm{Ins}(f_1, f_2)$ such that $\langle\alpha\rangle\,;\pi$ equals f and $\langle\alpha\rangle * \pi_{\mathrm{Ins}}$ equals α.

Example. Given an endofunctor $T : \mathbf{C} \to \mathbf{C}$, the category $\mathbf{Alg}(T)$ is the inserter from T to $\mathrm{Id}_\mathbf{C}$, and the category $\mathbf{Coalg}(T)$ is the inserter from $\mathrm{Id}_\mathbf{C}$ to T.

Definition (Coinserter Object)**.** Given two 1-cells $f_1, f_2 : A \to B$ of a 2-category, a coinserter from f_1 to f_2 is a 0-cell, $\mathrm{Coins}(f_1, f_2)$, along with 1-cell $\kappa : B \to \mathrm{Coins}(f_1, f_2)$ and 2-cell $\kappa_{\mathrm{Coins}} : f_1 \,;\kappa \Rightarrow f_2 \,;\kappa : A \to \mathrm{Coins}(f_1, f_2)$ that is *(co)universal*, meaning given any other 0-cell C with 1-cell $f : B \to C$ and 2-cell $\alpha : f_1 \,; f \Rightarrow f_2 \,; f : A \to C$ there exists a unique 1-cell $[\alpha] : \mathrm{Coins}(f_1, f_2) \to C$ such that $\kappa\,;[\alpha]$ equals f and $\kappa_{\mathrm{Coins}} *[\alpha]$ equals α.

Definition (Weighted Limit)**.** Let $\mathbf{I}$ be a 2-category conceptually describing a scheme, and let $D : \mathbf{I} \to \mathbf{C}$ be a 2-functor conceptually describing a diagram of scheme $\mathbf{I}$ in the 2-category $\mathbf{C}$. Furthermore, let $W : \mathbf{I} \to \mathbf{Cat}$ be a 2-functor conceptually describing a *weighting* of the diagram. A W-weighted *cone* of the diagram D, denoted $\lim_W D$, is a 0-cell L of $\mathbf{C}$ and a collection of 1-cells $\{\pi_w : L \to DI\}_{I \in \mathbf{I}, w \in WI}$ and 2-cells $\{\pi_\omega : \pi_w \Rightarrow \pi_{w'}\}_{I \in \mathbf{I}, \omega : w \to w' \in WI}$ that preserves identities and compositions, meaning $\pi_{id_w} = id_{\pi_w}$ and $\pi_{\omega\,;\,\omega'} = \pi_\omega \,; \pi_{\omega'}$, and is *natural*, meaning for all 1-cells $i : I \to I' \in \mathbf{I}$ each appropriate 1-cell $\pi_{(Wi)(w)}$ equals $\pi_w \,; Di$ and for all 2-cells $\iota : i \Rightarrow i' \in \mathbf{I}$ each appropriate 2-cell $\pi_{(W\iota)_\omega}$ equals $\pi_\omega * D\iota$. A W-weighted *limit* of a diagram D is a universal W-weighted cone of D.

Example. An inserter is a weighted limit as illustrated below:



Example. A comma object is a weighted limit. The scheme is $\bullet \to \bullet \leftarrow \bullet$ and the weighting is $1 \hookrightarrow (1 \overset{\downarrow}{\to} 2) \hookleftarrow 2$.

Definition. A weighted colimit is dual to a weighted limit: given a diagram $D : \mathbf{I} \to \mathbf{C}$ and weighting $W : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$ (the reason that W is contravariant here is complicated and very meta), a weighted colimit $\mathrm{colim}_W D$ in $\mathbf{C}$ is a weighted limit $\lim_W D^{\mathrm{op}}$ in $\mathbf{C}^{\mathrm{op}}$.

Example. Coinserter and cocomma objects are weighted colimits of the same weighting but on the opposite scheme as for inserter and comma objects.