Practice 11

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Definition (Multilinear Map)**.** A multilinear map from a list of commutative monoids $\langle A_1, 0_1, +_1\rangle, \ldots, \langle A_n, 0_n, +_n$ to a commutative monoid $\langle B, 0, +\rangle$ is a n-ary function $f : A_1 \times \cdots \times A_n \to B$ satisfying the following properties for any index $i \in \{1, \ldots, n\}$ and elements $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}, a_{i+1} \in A_{i+1}, \ldots, a_n \in A_n$:

$$f(a_1, \ldots, 0_i, \ldots, a_n) = 0 \quad \text{and} \quad \forall a_i, a'_i \in A_i.\ f(a_1, \ldots, a_i +_i a'_i, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) + f(a_i, \ldots, a'_i, \ldots, a_n)$$

That is, a multilinear map is an n-ary function such that, for every index i, fixing all the inputs *besides* the input for i results in a monoid homomorphism from $\langle A_i, 0_i, +_i\rangle$ to $\langle B, 0, +\rangle$. In particular, a unary multilinear map is simply a monoid homomorphism, and a nullary multilinear is simply a nullary function (i.e. an element of the codomain).

Exercise 1. Prove that **CommMon**, comprised of commutative monoids and multilinear maps with identities and composition inherited from **Set**, is a multicategory. In particular, demonstrate why the monoids must be commutative.

Exercise 2. Show that the internal monoids of **CommMon** bijectively correspond with semirings (defined in the previous homework).

Definition (Unit)**.** An object I with a multimorphism $\mathsf{unit} : [\,] \to I$ is called a unit object with a unit multimorphism if they form a tensor of the empty list. A multicategory has a unit if it has such an object and multimorphism.

Exercise 3. Prove that $\langle \mathbb{N}, 0, +\rangle$ is a unit object of **CommMon**. However, rather than showing the existence and uniqueness of $\mathsf{split}_{\vec{A};\varnothing;\vec{B}} f$ for arbitray lists of commutative monoids $\vec{A}$ and $\vec{B}$, for the sake of readability show this only for the case where $\vec{A}$ is a singleton list and $\vec{B}$ is empty.

Exercise 4. Prove that, for any given commutative monoids $\langle A, 1, *\rangle$ and $\langle B, 1, *\rangle$, the set of monoid homomorphisms from $\langle A, 1, *\rangle$ to $\langle B, 1, *\rangle$ is the underlying set of the left-exponential object $\langle A, 1, *\rangle \multimap \langle B, 1, *\rangle$ in **CommMon**. However, for the existence and uniqueness proof of λf, for the sake of readability only show this for the case where f is binary.

Exercise 5. Prove that, in **CommMon**, the underlying set of the tensor of $\langle A, 0, +\rangle$ and $\langle B, 0, +\rangle$ is the set ${}^{\mathbb{M}(A\times B)}/_{\approx}$, where $\approx$ is the least equivalence relation satisfying:

$$[\langle 0, b\rangle] \approx [\,] \qquad [\langle a + a', b\rangle] \approx [\langle a, b\rangle, \langle a', b\rangle] \qquad [\langle a, 0\rangle] \approx [\,] \qquad [\langle a, b + b'\rangle] \approx [\langle a, b\rangle, \langle a, b'\rangle]$$

$$\ell_1 \approx \ell'_1 \wedge \ell_2 \approx \ell'_2 \implies \ell_1 \mathbin{+\!\!+} \ell_2 \approx \ell'_1 \mathbin{+\!\!+} \ell'_2 \qquad\qquad (\ell \mathbin{+\!\!+} \ell' \approx \ell' \mathbin{+\!\!+} \ell)$$

Exercise 6. Prove that a unit of a *cartesian* multicategory is also a terminal object.