Assignment 2

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Exercise 1. Prove that every *bijective* relation-preserving function $f : \langle X, R\rangle \to \langle X, R\rangle$ is an isomorphism on $\langle X, R\rangle$ in $\mathbf{Rel}(2)$ if R is an antisymmetric and *connex* binary relation on X. Connexity means it satisfies the following property:

$$\forall x_1, x_2 \in X.\ R(x_1, x_2) \vee R(x_2, x_1)$$

In other words, all *unordered* pairs are related by R in some way. Hint: you will want a lemma that any connex relation is also reflexive. (Fun fact: this implies that the inverse of any bijective monotone function on $\mathbb{R}$ is also monotone.)

Exercise 2. Given a set X, its subsets bijectively correspond with the isomorphism-classes of its subobjects in $\mathbf{Set}$. Given an operation $+ : X \times X \to X$ on X, because monomorphisms in $\mathbf{Alg}(+ : 2)$ are precisely the morphisms with injective underlying functions, the isomorphism-classes of subobjects of $\langle X, +\rangle$ in $\mathbf{Alg}(+ : 2)$ bijectively correspond with a particular kind of subsets of X.

1. Give a property $\phi_+ \subseteq \mathbb{P}(X)$ of subsets of X.
2. Prove that every subset $\mathcal{X} \in \phi_+$ can be given an algebra $+_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ such that $\langle \mathcal{X}, +_{\mathcal{X}}\rangle \xhookrightarrow{\subseteq} \langle X, +\rangle$ is a subobject in $\mathbf{Alg}(+ : 2)$.
3. Prove that every subobject $\langle A, +_A\rangle \xhookrightarrow{i} \langle X, +\rangle$ in $\mathbf{Alg}(+ : 2)$ is isomorphic to $\langle \mathcal{X}, +_{\mathcal{X}}\rangle \xhookrightarrow{\subseteq} \langle X, +\rangle$ for some subset $\mathcal{X} \in \phi_+$. For this step of the proof, to reduce tedium, apply the fact that $\mathbf{Alg}(+ : 2)$, as a category of structured sets, *reflects isomorphisms*, meaning that a morphism in $\mathbf{Alg}(2)$ is an isomorphism if its underlying function is bijective.

Note: this process is easy to generalize to $\mathbf{Alg}(\Omega)$.

Definition. Given a (equivalence) relation $\approx \subseteq X \times X$ on a set X, the set $X/\!\approx$ is the "quotient" of X by $\approx$, which set-theoretically denotes the set of "equivalence classes" of $\approx$. The surjective function $\lambda x.[x]_\approx : X \to X/\!\approx$ maps each element of X to its "equivalence class" so that $[x]_\approx = [x']_\approx$ holds if $x \approx x'$ holds. Type-theoretically, a function f from $X/\!\approx$ to any set Y is simply a function $f : X \to Y$ such that $\forall x, x' \in X.\ x \approx x' \implies f(x) = f(x')$. For this class, we will use this type-theoretic convention for functions from quotients, as its much simpler than the set-theoertic convention and more consistent with the conventions for functions to subsets.

Exercise 3. Given a set X, its equivalence relations bijectively correspond with the isomorphism-classes of its quotient objects in $\mathbf{Set}$. Given an operation $+ : X \times X \to X$ on X, because epimorphisms in $\mathbf{Alg}(+ : 2)$ are precisely the morphisms with surjective underlying functions, the isomorphism-classes of quotient objects of $\langle X, +\rangle$ in $\mathbf{Alg}(+ : 2)$ bijectively correspond with a particular kind of equivalence relations on X.

1. Give a property $\psi_+ \subseteq \mathsf{Equiv}(X)$ of equivalence relations of X.
2. Prove that every equivalence relation $\approx \in \psi_+$ can be given an algebra $+_\approx : X/\!\approx \times X/\!\approx \to X/\!\approx$ such that $\langle X, +\rangle \xrightarrow{[\cdot]_\approx} \langle X/\!\approx, +_\approx\rangle$ is a quotient object in $\mathbf{Alg}(+ : 2)$.
3. Prove that every quotient object $\langle X, +\rangle \xrightarrow{e} \langle A, +_A\rangle$ in $\mathbf{Alg}(+ : 2)$ is isomorphic to $\langle X, +\rangle \xrightarrow{[\cdot]_\approx} \langle X/\!\approx, +_\approx\rangle$ for some equivalence relation $\approx \in \psi_+$. For this step of the proof, to reduce tedium, apply the fact that $\mathbf{Alg}(+ : 2)$, as a category of structured sets, *reflects isomorphisms*, meaning that a morphism in $\mathbf{Alg}(2)$ is an isomorphism if its underlying function is bijective.

Note: this process is easy to generalize to $\mathbf{Alg}(\Omega)$ provided every arity A in Ω is *projective*, meaning for any set X and total relation $R \subseteq A \times X$ there exists a function $f : A \to X$ such that $\forall a \in A.\ R(a, f(x))$. Every finite set is projective.