Assignment 7

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Exercise 1. Given a diagram $D : \mathbf{I} \to \mathbf{C}$ where $\mathbf{I}$ has a terminal object 1 and $\mathbf{C}$ has pullbacks, show that for every morphism $f : A \to D(1)$ there exists a diagram $D_f : \mathbf{I} \to \mathbf{C}$ and a collection of morphisms $\{f_i : D_f(i) \to D(i)\}_{i \in \mathbf{I}}$ such that f_1 equals f and for every morphism $m : i \to j$ in $\mathbf{I}$ the following is a pullback:

$$\begin{array}{ccc} D_f(i) & \xrightarrow{f_i} & D(i) \\ {\scriptstyle D_f(m)}\downarrow & & \downarrow{\scriptstyle D(m)} \\ D_f(j) & \xrightarrow[f_j]{} & D(j) \end{array}$$

To save a step in the proof, utilize Proposition 11.10(2) of *Abstract and Concrete Categories.*

Remark. The above process can in particular be used to construct the pullback of any finite cocone. Thus, just like there is a notion of pullback stability for classes of morphisms, there is also a notion of pullback stability for classes of cocones.

Definition. A pair of morphisms $p_1, p_2 : R \to A$ are a kernel pair if there exists a morphism $f : A \to B$ such that the following is a pullback:

$$\begin{array}{ccccc} & & R & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ A & & & & A \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f} & \\ & & B & & \end{array}$$

Definition. A triple $\langle p_1 : R \to A, p_2 : R \to A, c : A \to B \rangle$ is an exact sequence when $\langle p_1, p_2 \rangle$ is a kernal pair and c is a coequalizer of $\langle p_1, p_2 \rangle$.

Exercise 2. Show that for any coequalizer c of any kernel pair $\langle p_1, p_2 \rangle$, the following is a pullback

$$\begin{array}{ccccc} & & R & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ A & & & & A \\ & {\scriptstyle c}\searrow & & \swarrow{\scriptstyle c} & \\ & & B & & \end{array}$$

In other words, $\langle p_1, p_2 \rangle$ is specifically a kernel pair of c.

Exercise 3. Show that the underlying functor of $\mathbf{Alg}(2)$ reflects coequalizers, meaning that if $|c|$ is a coequalizer of $|f_1|$ and $|f_2|$ in $\mathbf{Set}$ then c is a coequalizer of f_1 and f_2 in $\mathbf{Alg}(2)$.

Exercise 4. Without using quotient types or sets of equivalence classes, show that every kernel pair in $\mathbf{Alg}(2)$ has a coequalizer and, using this construction, that the underlying functor of $\mathbf{Alg}(2)$ preserves exact sequences.

Definition. A category is regular if and only if it is a finitely complete category such that exact sequences are pullback stable and every kernel pair has a coequalizer (i.e. is part of some exact sequence).

Exercise 5. Show that $\mathbf{Alg}(2)$ is a regular category (understanding that we have already proven in class that $\mathbf{Alg}(2)$ is finitely complete). Hint: take advantage of the fact that $\mathbf{Set}$ is regular.