Transpositions and Adjunctions

Ross Tate

April 5, 2018

Definition (Transposition)**.** Given a pair of categories and pair of functors as in $\mathbf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{D}$, a transposition between L and R

- assigns to each L-costructured morphism $g : LC \to D$ an R-structured morphism $g^{\rightarrow} : C \to RD$
- and assigns to each R-structured morphism $f : C \to RD$ an L-costructured morphism $f^{\leftarrow} : LC \to D$
- such that the assignments are bijective, meaning $\forall g : LC \to D.\ (g^{\rightarrow})^{\leftarrow} = g$ and $\forall f : C \to RD.\ (f^{\leftarrow})^{\rightarrow} = f$,
- and natural, meaning $\forall f' : C \to C', g' : D \to D'$ we have $\forall g : LC' \to D.\ (Lf' \,;g\,;g')^{\rightarrow} = f' \,;g^{\rightarrow} \,;Rg'$ and $\forall f : C' \to RD.\ (f' \,;f\,;Rg')^{\leftarrow} = Lf' \,;f^{\leftarrow} \,;g'$.

Example. Given a subcategory $\mathbf{A} \overset{I}{\hookrightarrow} \mathbf{B}$ and a reflector $R : \mathbf{B} \to A$ with reflection arrows $\{r_B : B \to IRB\}_{B \in \mathbf{B}}$, we can build a transposition $R \dashv I$. For a $\mathbf{B}$-morphism $f : B \to IA$, define $f^{\leftarrow} : RB \to A$ as the $\mathbf{A}-$morphism that is uniquely induced by the reflection arrow r_B. For a $\mathbf{A}$-morphism $g : RB \to A$, define $g^{\rightarrow} : B \to IA$ as $r_B \,;Ig$.

Example. Given a category $\mathbf{C}$ and a set I, we can define the category $\mathbf{C}^I$ whose objects are I-indexed tuples of $\mathbf{C}$ objects and whose morphisms are I-indexed tuples of $\mathbf{C}$ morphisms, with the remaining structure defined in the obvious way. We can also define a functor $\Delta_I : \mathbf{C} \to \mathbf{C}^I$ that maps each object/morphism of $\mathbf{C}$ to the I-tuple simply comprised of I copies of the object/morphism. And if $\mathbf{C}$ has I-indexed products, we can define a functor $\prod_I : \mathbf{C}^I \to \mathbf{C}$ mapping each I-indexed tuple of $\mathbf{C}$ objects to their product and each I-indexed tuple of morphisms to the corresponding uniquely induced morphism between those products. These functors extend to a transposition $\Delta_I \dashv \prod_I$. To see why, note that an I-indexed source with domain C corresponds to an I-indexed tuple of morphisms with domain $\Delta_I(C)$. Consequently, for an I-indexed tuple of morphisms with domain $\Delta_I(C)$, i.e. a source $\{C \xrightarrow{g_i} C_i\}_{i \in I}$, define $(\{C \xrightarrow{g_i} C_i\}_{i \in I})^{\rightarrow} : C \to \prod_{i \in I} C_i$ as $\langle g_i \rangle_{i \in I}$. And for a $\mathbf{C}$-morphism $f : C \to \prod_{i \in I} C_i$, define $f^{\leftarrow} : \Delta_I(C) \to \{C_i\}_{i \in I}$ as $\{C \xrightarrow{f\,;\pi_i} C_i\}_{i \in I}$.

Example. If a category $\mathbf{C}$ has I-indexed coproducts, we can define a functor $\coprod_I : \mathbf{C}^I \to \mathbf{C}$ mapping each I-indexed tuple of $\mathbf{C}$ objects to their coproduct and each I-indexed tuple of morphisms to the corresponding uniquely induced morphism between those coproducts. These functors extend to a transposition $\coprod_I \dashv \Delta_I$. To see why, note that an I-indexed sink with codomain C corresponds to an I-indexed tuple of morphisms with codomain $\Delta_I(C)$. Consequently, for an I-indexed tuple of morphisms with codomain $\Delta_I(C)$, i.e. a sink $\{C_i \xrightarrow{f_i} C\}_{i \in I}$, define $(\{C_i \xrightarrow{f_i} C\}_{i \in I})^{\leftarrow} : \coprod_{i \in I} C_i \to C$ as $[f_i]_{i \in I}$. And for a $g : \coprod_{i \in I} C_i \to C$, define $g^{\rightarrow} : \{C_i\}_{i \in I} \to \Delta_I(C)$ as $\{C_i \xrightarrow{\kappa_i\,;g} C\}_{i \in I}$.

Example. Given a concrete category $\mathbf{A} \xrightarrow{U} \mathbf{X}$ and a free-object functor $F : \mathbf{X} \to U$ with universal structured arrows $\{\eta_X : X \to UFX\}_{X \in \mathbf{X}}$, we can build a transposition $F \dashv U$. For a $\mathbf{X}$-morphism $f : X \to UA$, define $f^{\leftarrow} : FB \to A$ as the $\mathbf{A}-$morphism that is uniquely induced by the universal structured arrow η_X. For a $\mathbf{A}$-morphism $g : FX \to A$, define $g^{\rightarrow} : X \to UA$ as $\eta_X \,;Ug$.

Definition (Adjunction)**.** Given a 2-category, an adjunction $\langle \eta, \varepsilon \rangle : \ell \dashv r : D \to C$ (where the order of D and C here is not a typo) is comprised of

- 0-cells C and D,
- 1-cells $\ell : C \to D$ (called the left adjoint) and $r : D \to C$ (called the right adjoint), and
- 2-cells $\eta : C \Rightarrow \ell \,; r : C \to C$ (called the unit) and $\varepsilon : r \,; \ell \Rightarrow D : D \to D$ (called the counit)
- such that both of the following compositions equal their respective identity 2-cell:

Example. Suppose we have a transposition between functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$. Then we build an adjunction $\langle \eta, \varepsilon \rangle : L \dashv R : \mathbf{D} \to \mathbf{C}$ by defining the natural transformation $\eta_C : C \to RLC$ as $(id_{LC})^{\rightarrow}$ and the natural transformation $\varepsilon_D : LRD \to D$ as $(id_{RD})^{\leftarrow}$.

Example. Suppose we have an adjunction $\langle \eta, \varepsilon \rangle : L \dashv R : \mathbf{D} \to \mathbf{C}$ in **Cat**. Then we can build a transposition by defining $g^{\rightarrow}$ for $g : LC \to D$ as $\varepsilon_C \,; Rg$ and defining $f^{\leftarrow}$ for $f : C \to RD$ as $Lf \,; \eta_D$.

Example. In adjunction in **Prost** is also known as a (monotone) Galois connection. A (monotone) Galois connection between two preorded sets $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ is a pair of relation-preserving functions $F : \langle A, \leq \rangle \to \langle B, \leq \rangle$ and $G : \langle B, \leq \rangle \to \langle A, \leq \rangle$ such that $\forall a \in A, b \in B.\ F(a) \leq b \iff a \leq G(b)$.

Definition (Equivalence)**.** An equivalence is an adjunction in which η and ε are both 2-isomorphisms. Two objects of a 2-category are said to be equivalent if there exists an equivalence between them. In particular, two categories are said to be equivalent if there exists an equivalence between them in **Cat**.

Example. Assuming the axiom of choice, every preorder $\langle X, \leq \rangle$ is equivalent in **Prost** to its antisymmetric quotient $\langle {}^{X}\!/\!\approx, \leq \rangle$.

Example. The category of finite vector spaces and linear maps is equivalent to **Mat**.