Order of quantifiers

What do the following mean?

(a) $\forall x \exists y \text{ s.t. } x \text{ loves } y$ means everybody loves somebody
(b) $\exists y \forall x \text{ s.t. } x \text{ loves } y$ means there is somebody who is loved by everybody.

pf of (a): choose arb. $x$, let $y = x$'s mother. WTS $x$ loves $y$.
well, $y$ gave birth to $x$, so $x$ must love $y$.

pf of (b): let $y = x$'s mother
let $y = \text{raymond}$. choose arb. $x$. Then $x$ loves Raymond because everybody loves Raymond. $\checkmark$
Negating quantified statements

- **Defn**: \(|A| \geq |B|\) if there is a function \(f : A \to B\) that is surjective
- **Defn**: \(f : A \to B\) is surjective if every \(y \in B\), there is some \(x \in A\) with \(f(x) = y\)
- **Defn**: \(y\) is in the image of \(f\) if there is some \(x \in A\) with \(f(x) = y\)

Putting these together (symbolically): \(|A| \geq |B|\) if:

\[
\forall f : A \to B, \exists x \in A \text{ s.t. } f(x) = y.
\]

Question: what does \(|A| \neq |B|\) mean?

- In this style
- In this style

Reminder:
- \(\forall x, P(x)\) is false if \(\exists x, P(x)\) is false.
- \(\exists x, P(x)\) is false if \(\forall x, \neg P(x)\)

Answer:
- \(|A| \neq |B|\) means there is not \(f : A \to B\) is surjective.
  - i.e. \(\forall f : A \to B\), \(f\) is not surjective.
  - \(f\) is not surjective if it is false that every \(y\) is in image of \(f\).
    - i.e. \(\exists y \in B\) that is not in im. of \(f\).
    - \(y\) is not in image of \(f\) means that there is no \(x\) that maps to \(y\).
      - i.e. \(\forall x \in A\), \(f(x) \neq y\).

So \(|A| \neq |B|\) means:

\[
\forall f : A \to B, \exists y \in B, \forall x \in A, f(x) \neq y.
\]
Relations

**function** \[ f(x) \]

\[
\begin{array}{c|c}
 x & f(x) \\
 \hline
 0 & a \\
 1 & b \\
 2 & c \\
\end{array}
\]

**relation**

<table>
<thead>
<tr>
<th>Name</th>
<th>Salary</th>
<th>Title</th>
<th>Emp. Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike</td>
<td>a lot</td>
<td>A</td>
<td>2013</td>
</tr>
<tr>
<td>Mike</td>
<td>a little</td>
<td>B</td>
<td>2015</td>
</tr>
</tbody>
</table>

Set of rows, each row has Name, Salary, ...

\[
\{(\text{Mike}, \text{a lot}, \text{A}, 2013)\}, \quad \{(\text{Mike}, \text{a little}, \text{B}, 2015)\}
\]

\[ \subseteq \text{Names} \times \text{Salaries} \times \text{Titles} \times \text{Dates} \]

**Def:** A relation \( R \) on sets \( A, B, C, ... \) is a subset of \( A \times B \times C \times ... \)
Binary relations

**Def** : A binary relation on a set $A$ is a subset of $A \times A$.

**Example** : "is-friend-of" is a relation on the set of people.

\[
\{(Alice, Bob), (Bob, Alice), (Bob, Chuck), (Chuck, Dave)\}
\]

- Bob is a friend of Chuck
- Chuck is not a friend of Bob

**Notation** : if $R$ is a bin. rel. on $A$, we write $xRy$ to mean $(x,y) \in R$.

**Example** : "=" is a relation on any set.

- "=" for $f:A \to A$ is a relation on set of all $f^a$.
- "=" for sets is a set of all sets.
- "1:1 = 1:1" is a relation on set of all sets.
- "≠" is a binary rel.
- "≤" is an order rel.

$3 \leq 5$ means $(3,5) \in \leq$
Properties of "equivalence" $R$ on a set $A$ we expect "equivalence" $\sim$ to satisfy:

- **Reflexivity:** $\forall x \in A$, $x R x$

- **Symmetry:** $\forall x, y \in A$, if $x R y$ then $y R x$.

- **Transitivity:** $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x R z$. 